

Linear-size sparsifiers for sums of seminorms

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Abstract

Let $N_1, \dots, N_m$ be seminorms on $\mathbb{R}^n$, and write $N = \sum_{i=1}^m N_i$. We prove that, for every $\varepsilon \in (0, 1/2]$, there are nonnegative weights $w_1, \dots, w_m$, at most

$$O\left(\frac{d}{\varepsilon^2} \log \frac{1}{\varepsilon}\right)$$

of which are nonzero, such that

$$(1 - \varepsilon)N(x) \leq \sum_{i=1}^m w_i N_i(x) \leq (1 + \varepsilon)N(x) \quad \text{for every } x \in \mathbb{R}^n.$$

Here $d = \dim(\mathbb{R}^n / \ker N) \leq n$ is the effective dimension of the sum. This removes the dimension-dependent polylogarithmic losses in the previous general bounds for sums of seminorms. As consequences, sums of normalized symmetric submodular functions admit sparsifiers of the same size, with the natural effective dimension, and every weighted hypergraph on n vertices has a cut sparsifier with $O((n - \kappa)\varepsilon^{-2} \log(1/\varepsilon))$ hyperedges, where κ is the number of connected components of its positive-weight 2-section.

At a high level, we represent seminorms as support functions of compact centrally symmetric convex sets and extend the recent coefficient-space argument of Reis and Rothvoss from sums of segments to arbitrary Minkowski sums. Convexity of volume under Minkowski erosion supplies the required volumetric estimate, and partial coloring then eliminates a constant fraction of the summands at each step.

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1 Introduction

Consider seminorms $N_1, \dots, N_m : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and their sum

$$N(x) = \sum_{i=1}^m N_i(x). \tag{1}$$

The sparsification problem asks whether most of the summands can be discarded after reweighting while preserving N simultaneously on all of $\mathbb{R}^n$. More precisely, one seeks a nonnegative vector $w \in \mathbb{R}_+^m$ with small support for which

$$\left| N(x) - \sum_{i=1}^m w_i N_i(x) \right| \leq \varepsilon N(x) \quad \text{for every } x \in \mathbb{R}^n. \tag{2}$$

The familiar special case $N_i(x) = |\langle a_i, x \rangle|$ is row sparsification in ℓ_1 . Graph cut sparsification is obtained by taking a_i to be signed edge-incidence vectors.

Jambulapati, Lee, Liu, and Sidford [JLLS23] proved that arbitrary sums of seminorms admit sparsifiers with

$$O\left(\varepsilon^{-2}n \log(n/\varepsilon)(\log n)^{5/2}\right)$$

nonzero weights. Their result also gives an efficient construction under a standard rounding assumption. For one-dimensional summands, Talagrand's classical theorem [Tal90] gives $O(\varepsilon^{-2}n \log n)$ rows. Very recently, Reis and Rothvoss [RR26] removed the dimension-dependent logarithm and proved the bound $O(\varepsilon^{-2}n \log(1/\varepsilon))$ for this case.

Our main result gives the latter bound for arbitrary seminorms. It is convenient to state it in terms of the dimension on which the sum is a genuine norm.

Theorem 1.1. *Let $N_1, \dots, N_m$ be seminorms on $\mathbb{R}^n$, let $N = \sum_{i=1}^m N_i$, and set*

$$d = \dim(\mathbb{R}^n / \ker N).$$

For every $\varepsilon \in (0, 1/2]$, there is a vector $w \in \mathbb{R}_+^m$ such that

$$|\text{supp}(w)| \leq C \frac{d}{\varepsilon^2} \log \frac{1}{\varepsilon}$$

and

$$(1 - \varepsilon)N(x) \leq \sum_{i=1}^m w_i N_i(x) \leq (1 + \varepsilon)N(x) \quad \text{for every } x \in \mathbb{R}^n.$$

Here $C > 0$ is a universal constant.

The dependence on d is necessary already for graph cuts, up to the $\log(1/\varepsilon)$ factor; see, for example, the lower bounds discussed in [AKW14]. Theorem 1.1 is existential. The Reis–Rothvoss framework is also existential in the ℓ_1 setting and does not yield a polynomial-time construction [RR26].

Proof idea. Every seminorm is the support function of a compact centrally symmetric convex set. Thus, if C_i is the support body of N_i , then N is the support function of the Minkowski sum $C = C_1 + \dots + C_m$. Condition (2) is equivalent to

$$(1 - \varepsilon)C \subseteq \sum_{i=1}^m w_i C_i \subseteq (1 + \varepsilon)C.$$

Reis and Rothvoss prove this inclusion when every C_i is a segment. Their proof has two parts. A volumetric argument shows that the set of small simultaneous perturbations of the coefficients has large volume, and a partial-coloring theorem then finds a perturbation that sets a constant fraction of the coefficients to zero. Iterating the step gives the desired sparsity.

Only one part of the volumetric argument uses that the summands are segments: the assertion that volume is separately convex under Minkowski erosion by the summands. We prove the following general replacement:

$$t \mapsto \text{Vol}(K \ominus tL) \quad \text{is convex on } \mathbb{R}_+ \tag{3}$$

for arbitrary convex bodies K, L . For full-dimensional L , the derivative of this function is minus the anisotropic perimeter of the eroded body. That perimeter is a mixed volume, hence decreases under set inclusion. Lower-dimensional L follow by approximation. This one-variable statement implies separate convexity when the summands are arbitrary support bodies, after which the coefficient-space proof of [RR26] applies without change. We include all steps needed for the reduction.

Applications. The Lovász extension of a nonnegative normalized symmetric submodular function is a seminorm [Lov83]. We therefore obtain a sparsifier of size $O(n\varepsilon^{-2}\log(1/\varepsilon))$ for every decomposable symmetric submodular function, improving the earlier general bound from [JLLS23]. A hyperedge cut is one such function. Equivalently, its Lovász extension is the range seminorm on the hyperedge. This yields a weighted hypergraph cut sparsifier with $O(n\varepsilon^{-2}\log(1/\varepsilon))$ hyperedges. Prior polynomial-time constructions use $O(n\varepsilon^{-2}\log n)$ hyperedges [CKN20]; our conclusion concerns existence rather than efficient construction.

Organization. Section 2 records the convex-geometric translation and the coefficient-space results from [RR26]. Section 3 proves (3) and a random inclusion theorem for Minkowski sums. In Section 4, we establish the large-volume perturbation body and prove Theorem 1.1. Section 5 gives the submodular and hypergraph consequences.

2 Preliminaries

All convex sets in this paper are subsets of a finite-dimensional Euclidean space. A *convex body* is a compact convex set with nonempty interior in the ambient space. We allow other compact convex sets to be lower-dimensional. For a compact convex set K , its support function is

$$h_K(x) = \max_{y \in K} \langle y, x \rangle.$$

Support functions characterize compact convex sets, and satisfy $h_{K+L} = h_K + h_L$ and $h_{tK} = th_K$ for $t \geq 0$. In particular, $K \subseteq L$ if and only if $h_K \leq h_L$ pointwise.

For compact convex sets K, L , with L nonempty, define their Minkowski erosion (also called Minkowski subtraction) by

$$K \ominus L = \{x : x + L \subseteq K\}.$$

This is a compact convex set, possibly empty, and

$$(K \ominus L) \ominus M = K \ominus (L + M). \tag{4}$$

If K is convex and $0 \leq t \leq 1$, then

$$K \ominus tK = (1 - t)K. \tag{5}$$

We use Vol_k for k -dimensional Lebesgue measure, with the convention $\text{Vol}_0(\{0\}) = 1$.

2.1 Support bodies of seminorms

Let N be a seminorm on $\mathbb{R}^n$. Its kernel is a linear subspace. If $E = (\ker N)^\perp$, then the restriction of N to E is a norm and $N(x) = N(P_E x)$, where P_E is orthogonal projection onto E .

Lemma 2.1. *For every seminorm N on $\mathbb{R}^n$, the set*

$$C_N = \{y \in \mathbb{R}^n : \langle y, x \rangle \leq N(x) \text{ for every } x \in \mathbb{R}^n\}$$

is compact, convex, centrally symmetric, and contained in $(\ker N)^\perp$. Moreover,

$$N(x) = h_{C_N}(x) \quad \text{for every } x \in \mathbb{R}^n.$$

If N is a norm on a Euclidean space E , then C_N has nonempty interior in E .

Proof. Convexity, closedness, and central symmetry follow directly from the definition. If $z \in \ker N$ and $y \in C_N$, then $|\langle y, z \rangle| \leq N(z) = 0$, so $C_N \subseteq (\ker N)^\perp$. On that orthogonal complement, equivalence of finite-dimensional norms gives $N(x) \leq R\|x\|_2$ for some $R < \infty$. Testing the defining inequality in the direction $x = y$ shows $\|y\|_2 \leq R$ for $y \in C_N$. Hence C_N is compact.

The inequality $h_{C_N} \leq N$ is part of the definition. For the reverse inequality, fix x . If $x \in \ker N$, both sides vanish. Otherwise, apply Hahn–Banach to the linear functional on $\text{span}\{x\}$ that maps tx to $tN(x)$, dominated by N . Its extension has the form $z \mapsto \langle y, z \rangle$ for some $y \in C_N$ and satisfies $\langle y, x \rangle = N(x)$. Thus $h_{C_N}(x) \geq N(x)$. Finally, when N is a norm, norm equivalence also gives $N(x) \geq r\|x\|_2$ for some $r > 0$, and therefore $rB_2 \subseteq C_N$. $\square$

For seminorms $N_1, \dots, N_m$, let $C_i = C_{N_i}$. With $N = \sum_i N_i$, we have

$$N = h_C, \quad C = \sum_{i=1}^m C_i. \tag{6}$$

If $E = (\ker N)^\perp$, then every $C_i \subseteq E$, and C is a centrally symmetric convex body in the d -dimensional space E . Indeed, $\ker N = \bigcap_i \ker N_i$ because the summands are nonnegative.

2.2 Two coefficient-space theorems

We record the two results from Reis and Rothvoss that drive the support reduction. For $A \subseteq \mathbb{R}^m$ and $S \subseteq [m]$, write

$$A_S = \{x_S \in \mathbb{R}^S : (x_S, 0_{[m] \setminus S}) \in A\}$$

for its coordinate section.

Theorem 2.2 (Reis–Rothvoss symmetrizer theorem). *Let $p, \eta \in (0, 1/2]$, and suppose $m \geq \log_2(1/p)/\eta^2$. Let $A \subseteq [-1, 1]^m$ be a convex body such that $[-\eta, \eta]^m \subseteq A$ and*

$$\text{Vol}_{|S|}(A_S) \geq p 2^{|S|} \quad \text{for every } S \subseteq [m].$$

Then

$$\text{Vol}_m(A \cap (-A)) \geq 2^{-5m}.$$

This is Theorem 16 of [RR26]. We also use their partial-coloring statement, which is a volume formulation of a result developed in [RR23].

Theorem 2.3 (Reis–Rothvoss partial coloring). *For every $c > 0$, there is a constant $s = s(c) > 0$ with the following property. If $Q \subseteq [-1, 1]^m$ is a centrally symmetric convex body and $\text{Vol}_m(Q) \geq c^m$, then there is $z \in sQ \cap [-1, 1]^m$ such that*

$$|\{i \in [m] : |z_i| = 1\}| \geq m/2.$$

This is Theorem 7 of [RR26]. Only the existence of a universal constant $s = s(2^{-5})$ will be relevant below.

3 Minkowski erosion and random sums

The main new geometric input is the following extension of the segment erosion lemma used in [RR26].

Lemma 3.1 (Convexity of erosion volume). *Let $K, L \subseteq \mathbb{R}^d$ be compact convex sets, with L nonempty. Set $\text{Vol}_d(\emptyset) = 0$. Then*

$$f(t) = \text{Vol}_d(K \ominus tL), \quad t \geq 0,$$

is convex.

Proof. If K is lower-dimensional, then $f \equiv 0$, so assume that K is full-dimensional. We first suppose that L is full-dimensional. Translating L by a vector a translates $K \ominus tL$ by $-ta$ and does not change its volume, so we may assume $0 \in \text{int}(L)$. Let

$$r = r(K; L) = \sup\{t \geq 0 : K \ominus tL \neq \emptyset\}$$

be the relative inradius, and write $K_t = K \ominus tL$ for $0 \leq t \leq r$. For $t < r$, the set K_t is full-dimensional.

Recall that the anisotropic perimeter of a convex body A relative to L is

$$P_L(A) = dV(A[d-1], L),$$

where V denotes mixed volume. The inner-parallel-body first variation formula gives

$$\frac{d}{dt} \text{Vol}_d(K_t) = -P_L(K_t), \quad 0 < t < r. \quad (7)$$

See, for example, Proposition 2.6 of [RSG22] or Section 7.5 of [Sch14]. If $0 \leq t < u < r$, then $K_u \subseteq K_t$. Monotonicity of mixed volumes in each argument therefore gives

$$P_L(K_u) \leq P_L(K_t).$$

Consequently, the derivative in (7) is nondecreasing, and f is convex on $[0, r)$. At $t = r$, K_r has empty interior and hence zero d -dimensional volume; otherwise a slightly larger translate of L would fit in K . Extending f by zero on $[r, \infty)$ preserves convexity because its left derivatives are nonpositive and the right derivative after r is zero.

It remains to remove the full-dimensionality assumption on L . Let B be the Euclidean unit ball and $L_\delta = L + \delta B$ for $\delta > 0$. The functions

$$f_\delta(t) = \text{Vol}_d(K \ominus tL_\delta)$$

are convex by the first part. By associativity of erosion,

$$K \ominus tL_\delta = (K \ominus tL) \ominus t\delta B. \quad (8)$$

As $\delta \downarrow 0$, the sets on the right increase to the interior of $K \ominus tL$ when that set is full-dimensional; their volumes converge to its volume. If $K \ominus tL$ is lower-dimensional, both sides of the desired volume limit are zero. Hence $f_\delta(t) \rightarrow f(t)$ pointwise for every $t \geq 0$. A pointwise limit of finite convex functions is convex, which completes the proof. $\square$

Corollary 3.2 (Separate convexity). *Let $K, L_1, \dots, L_m \subseteq \mathbb{R}^d$ be compact convex sets, with each L_i nonempty. The function*

$$F(t) = \text{Vol}_d\left(K \ominus \sum_{i=1}^m t_i L_i\right), \quad t \in \mathbb{R}_+^m,$$

is convex in each coordinate separately.

Proof. Fix all coordinates except t_i and use associativity (4) to write the displayed erosion as $K' \ominus t_i L_i$ for a fixed compact convex set K' . If K' is empty, the function is zero. Otherwise, Lemma 3.1 applies. $\square$

We next generalize the random-zonotope inclusion theorem from [RR26] to arbitrary Minkowski summands.

Theorem 3.3 (Random Minkowski-sum inclusion). *Let $K \subseteq \mathbb{R}^d$ be a centrally symmetric convex body, and let $L_1, \dots, L_m \subseteq \mathbb{R}^d$ be nonempty compact centrally symmetric convex sets. For $t \in \mathbb{R}_+^m$, write $L_t = \sum_{i=1}^m t_i L_i$. If $X \in \mathbb{R}_+^m$ has independent integrable coordinates, then*

$$\Pr[L_X \subseteq K] \geq \frac{\text{Vol}_d(K \ominus L_{\mathbb{E}X})}{\text{Vol}_d(K)}.$$

Proof. Set $F(t) = \text{Vol}_d(K \ominus L_t)$. By Corollary 3.2, F is separately convex. Repeated one-dimensional Jensen inequalities, using independence of the coordinates, give

$$\mathbb{E}F(X) \geq F(\mathbb{E}X). \quad (9)$$

Every erosion $K \ominus L_X$ is contained in K and therefore has volume at most $\text{Vol}_d(K)$. Moreover, positive volume implies that the erosion is nonempty, which for centrally symmetric K and L_X is equivalent to $L_X \subseteq K$. It follows that

$$\Pr[L_X \subseteq K] \text{Vol}_d(K) \geq \mathbb{E}F(X) \geq F(\mathbb{E}X),$$

where the last inequality is (9). $\square$

4 Sparsification

We first show that the body of coefficient perturbations preserving a Minkowski sum has large volume. This is the point at which Theorem 3.3 enters the Reis–Rothvoss proof.

Let $d \geq 1$. Let $C_1, \dots, C_m$ be nonempty compact centrally symmetric convex sets in $\mathbb{R}^d$, and suppose $C = \sum_{i=1}^m C_i$ is full-dimensional. For $t \in \mathbb{R}_+^m$, write $C_t = \sum_i t_i C_i$.

Proposition 4.1 (Large perturbation body). *Fix $\eta \in (0, 1/2]$, and define*

$$A_\eta = \{z \in [-1, 1]^m : C_{\mathbf{1}+z} \subseteq (1 + \eta)C\}.$$

Then A_η is convex, contains $[-\eta, \eta]^m$, and for every $S \subseteq [m]$,

$$\text{Vol}_{|S|}((A_\eta)_S) \geq \left(\frac{\eta}{2}\right)^d 2^{|S|}.$$

Consequently, if

$$m \geq \frac{2d \log_2(1/\eta)}{\eta^2}, \tag{10}$$

then the centrally symmetric convex body

$$Q_\eta = A_\eta \cap (-A_\eta) = \{z \in [-1, 1]^m : (1 - \eta)C \subseteq C_{\mathbf{1}+z} \subseteq (1 + \eta)C\}$$

satisfies $\text{Vol}_m(Q_\eta) \geq 2^{-5m}$.

Proof. Taking support functions gives

$$A_\eta = \left\{ z \in [-1, 1]^m : \sum_{i=1}^m z_i h_{C_i}(x) \leq \eta \sum_{i=1}^m h_{C_i}(x) \text{ for every } x \in \mathbb{R}^d \right\}. \tag{11}$$

Thus A_η is an intersection of halfspaces and is convex. The same display shows $[-\eta, \eta]^m \subseteq A_\eta$.

Fix $S \subseteq [m]$. Let X_i be uniform on $[0, 2]$ for $i \in S$ and set $X_i = 1$ deterministically for $i \notin S$. The coordinates are independent and $\mathbb{E}X = \mathbf{1}$. Theorem 3.3, applied with target $(1 + \eta)C$, yields

$$\begin{aligned} \frac{\text{Vol}_{|S|}((A_\eta)_S)}{2^{|S|}} &= \Pr[C_X \subseteq (1 + \eta)C] \\ &\geq \frac{\text{Vol}_d((1 + \eta)C \ominus C)}{\text{Vol}_d((1 + \eta)C)} = \left(\frac{\eta}{1 + \eta}\right)^d \geq \left(\frac{\eta}{2}\right)^d, \end{aligned}$$

where (5) gives the equality.

Intersecting (11) with its negative shows that $Q_\eta = A_\eta \cap (-A_\eta)$ has the claimed two-sided inclusion description. Set $p = (\eta/2)^d$. Since $\eta \leq 1/2$,

$$\log_2(1/p) = d \log_2(2/\eta) \leq 2d \log_2(1/\eta).$$

Under (10), all hypotheses of Theorem 2.2 hold, and hence $\text{Vol}_m(Q_\eta) \geq 2^{-5m}$. $\square$

We now iterate a partial coloring. The following elementary estimate will control the accumulated error.

Lemma 4.2. Let $\phi(u) = \log_2(1/u)/u^2$ for $u \in (0, 1/2]$. For every $a \geq 1$ and $u \in (0, 1/2]$,

$$a\phi(u) \geq \phi(ua^{-1/4}).$$

Proof. Writing $q = \log_2 a \geq 0$ gives

$$\frac{\phi(ua^{-1/4})}{\phi(u)} = a^{1/2} \left(1 + \frac{q}{4 \log_2(1/u)} \right) \leq a^{1/2} \left(1 + \frac{q}{4} \right) \leq a.$$

The last inequality is $1 + q/4 \leq 2^{q/2}$, which follows from convexity of the right-hand side and its derivative at $q = 0$. $\square$

Proof of Theorem 1.1. If $d = 0$, then $N \equiv 0$ and the zero vector of weights suffices. Otherwise, work in the d -dimensional space $E = (\ker N)^\perp$. Let C_i be the support body of N_i in E , as in Lemma 2.1, so that $C = \sum_i C_i$ is full-dimensional and (6) holds. Zero summands may be discarded at the outset.

Let $s = s(2^{-5})$ be the constant in Theorem 2.3. Choose a sufficiently large universal constant C_0 , depending only on s , and set

$$M = C_0 d \phi(\varepsilon) = C_0 \frac{d}{\varepsilon^2} \log_2 \frac{1}{\varepsilon}.$$

If $m \leq M$, take every weight equal to one. Suppose $m > M$.

Starting from $w^{(0)} = \mathbf{1}$, we construct nonnegative weight vectors $w^{(t)}$. Write

$$S_t = \text{supp}(w^{(t)}), \quad m_t = |S_t|, \quad C^{(t)} = \sum_{i \in S_t} w_i^{(t)} C_i.$$

As long as $m_t > M$, let $\eta_t \in (0, 1/2]$ be the unique solution of

$$m_t = 2d\phi(\eta_t). \tag{12}$$

Indeed, ϕ is continuous and strictly decreasing from infinity to 4 on $(0, 1/2]$. Moreover, $m_t > M$ and Lemma 4.2, applied with $a = C_0/2$, give

$$\eta_t \leq \varepsilon(C_0/2)^{-1/4}.$$

Thus the choice of C_0 ensures existence, $\eta_t \leq \varepsilon$, and $s\eta_t < 1/2$. Apply Proposition 4.1 to the active summands $w_i^{(t)} C_i$, $i \in S_t$. It gives a symmetric convex body

$$Q_t = \left\{ z \in [-1, 1]^{S_t} : (1 - \eta_t)C^{(t)} \subseteq \sum_{i \in S_t} (1 + z_i)w_i^{(t)} C_i \subseteq (1 + \eta_t)C^{(t)} \right\}$$

with $\text{Vol}_{m_t}(Q_t) \geq 2^{-5m_t}$. Theorem 2.3 supplies $z^{(t)} \in sQ_t \cap [-1, 1]^{S_t}$ for which at least $m_t/2$ coordinates have absolute value one. Since Q_t is symmetric, we may replace $z^{(t)}$ by its negative and arrange that at least $m_t/4$ coordinates equal -1 . Update

$$w_i^{(t+1)} = (1 + z_i^{(t)})w_i^{(t)} \quad (i \in S_t),$$

and keep all other weights zero. The weights remain nonnegative and

$$m_{t+1} \leq \frac{3}{4}m_t. \quad (13)$$

Moreover, because $z^{(t)} \in sQ_t$, the support-function description of Q_t gives

$$(1 - s\eta_t)C^{(t)} \subseteq C^{(t+1)} \subseteq (1 + s\eta_t)C^{(t)}. \quad (14)$$

Let T be the first index for which $m_T \leq M$. It remains to bound the accumulated error. From (13) and $m_{T-1} > M$, for $0 \leq t < T$ we have

$$m_t > M \left(\frac{4}{3}\right)^{T-1-t}.$$

Combining this with (12) and the definition of M gives

$$\phi(\eta_t) > \frac{C_0}{2}\phi(\varepsilon) \left(\frac{4}{3}\right)^{T-1-t}.$$

Lemma 4.2 and monotonicity of ϕ imply

$$\eta_t \leq \varepsilon \left(\frac{C_0}{2}\right)^{-1/4} \left(\frac{3}{4}\right)^{(T-1-t)/4}.$$

We choose C_0 large enough that

$$\sum_{t=0}^{T-1} s\eta_t \leq s\varepsilon \left(\frac{C_0}{2}\right)^{-1/4} \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^{j/4} \leq \frac{\varepsilon}{2}. \quad (15)$$

Iterating (14) and using (15),

$$\begin{aligned} \prod_{t < T} (1 - s\eta_t) &\geq 1 - \sum_{t < T} s\eta_t \geq 1 - \varepsilon, \\ \prod_{t < T} (1 + s\eta_t) &\leq \exp\left(\sum_{t < T} s\eta_t\right) \leq e^{\varepsilon/2} \leq 1 + \varepsilon. \end{aligned}$$

The final inequality holds for $0 \leq \varepsilon \leq 1/2$. We conclude that

$$(1 - \varepsilon)C \subseteq C^{(T)} \subseteq (1 + \varepsilon)C.$$

Taking support functions and using (6) proves the uniform seminorm approximation. Finally, $|\text{supp}(w^{(T)})| = m_T \leq M$, as required. $\square$

Remark 4.3. The proof uses no description or evaluation oracle for the summands. It therefore establishes existence for completely arbitrary finite-dimensional seminorms. Finding the partial coloring requires a separation oracle for a body defined by uniform comparison of two sums of seminorms. We do not assert that such separation, or the resulting sparsifier, can be computed in polynomial time in a general oracle model.

5 Symmetric submodular functions and hypergraph cuts

Let V be a finite ground set of size n . A function $f : 2^V \rightarrow \mathbb{R}_+$ is *submodular* if

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (A, B \subseteq V),$$

and *symmetric* if $f(S) = f(V \setminus S)$ for every $S \subseteq V$. We call f *normalized* when $f(\emptyset) = 0$; symmetry then also gives $f(V) = 0$.

For a normalized symmetric function, its Lovász extension may be written, up to immaterial choices at threshold ties, as

$$\bar{f}(x) = \int_{-\infty}^{\infty} f(\{v \in V : x_v \geq t\}) dt. \quad (16)$$

The integral is finite because the threshold set is V below the minimum coordinate and empty above the maximum coordinate.

Lemma 5.1. *If $f : 2^V \rightarrow \mathbb{R}_+$ is normalized, symmetric, and submodular, then $\bar{f}$ is a seminorm on $\mathbb{R}^V$. It vanishes on constant vectors and satisfies*

$$\bar{f}(\mathbf{1}_S) = f(S) \quad (S \subseteq V).$$

Moreover, the Lovász extension is linear in the set function.

Proof. The Lovász extension of a submodular function is convex [Lov83]. Formula (16) shows positive homogeneity. Symmetry of f , followed by the change of variables $t \mapsto -t$, gives $\bar{f}(-x) = \bar{f}(x)$; threshold ties affect only a measure-zero set of t . A convex, positively homogeneous, even function is a seminorm. The remaining assertions follow directly from (16). $\square$

Corollary 5.2 (Symmetric submodular sparsification). *Let $f_1, \dots, f_m : 2^V \rightarrow \mathbb{R}_+$ be normalized symmetric submodular functions, set $F = \sum_i f_i$, and let*

$$d_F = n - \dim \left(\bigcap_{i=1}^m \ker \bar{f}_i \right) \leq n - 1.$$

For every $\varepsilon \in (0, 1/2]$, there are nonnegative weights $w_1, \dots, w_m$ with

$$|\text{supp}(w)| = O\left(\frac{d_F}{\varepsilon^2} \log \frac{1}{\varepsilon}\right)$$

such that

$$(1 - \varepsilon)F(S) \leq \sum_{i=1}^m w_i f_i(S) \leq (1 + \varepsilon)F(S) \quad \text{for every } S \subseteq V.$$

In fact, the same weights approximate the Lovász extension of F on every vector in $\mathbb{R}^V$.

Proof. Apply Theorem 1.1 to the seminorms $\bar{f}_i$ from Lemma 5.1. Every $\bar{f}_i$ vanishes on the constant vectors, so $d_F \leq n - 1$. Evaluate the resulting inequality at $\mathbf{1}_S$. $\square$

The effective dimension in Corollary 5.2 can be smaller than $n - 1$ if the common kernel of the Lovász extensions is larger than the constants. The conclusion applies, for example, to sums of graph connectivity functions, hypergraph cut functions, and matroid connectivity functions.

We spell out the hypergraph case. Let $H = (V, \mathcal{E}, c)$ be a weighted hypergraph, where $c_e \geq 0$ for $e \in \mathcal{E}$. Its cut function is

$$\delta_H(S) = \sum_{e \in \mathcal{E}: e \cap S \neq \emptyset, e \setminus S \neq \emptyset} c_e.$$

For each hyperedge e , define the range function

$$R_e(x) = \max_{u \in e} x_u - \min_{v \in e} x_v$$

and the weighted range seminorm

$$N_e(x) = c_e R_e(x). \tag{17}$$

It is the Lovász extension of the cut indicator of e , and $N_e(\mathbf{1}_S) = c_e$ exactly when e crosses S .

Corollary 5.3 (Hypergraph cut sparsification). *For every weighted hypergraph $H = (V, \mathcal{E}, c)$ on n vertices, let $\kappa(H)$ be the number of connected components, including isolated vertices, of the graph that joins $u, v \in V$ when some positive-weight hyperedge contains both. For every $\varepsilon \in (0, 1/2]$, there is a weighted subhypergraph $H' = (V, \mathcal{E}', c')$ with*

$$|\mathcal{E}'| = O\left(\frac{n - \kappa(H)}{\varepsilon^2} \log \frac{1}{\varepsilon}\right)$$

such that

$$(1 - \varepsilon)\delta_H(S) \leq \delta_{H'}(S) \leq (1 + \varepsilon)\delta_H(S) \quad \text{for every } S \subseteq V.$$

The same reweighting satisfies the stronger continuous inequality

$$(1 - \varepsilon) \sum_{e \in \mathcal{E}} N_e(x) \leq \sum_{e \in \mathcal{E}'} c'_e R_e(x) \leq (1 + \varepsilon) \sum_{e \in \mathcal{E}} N_e(x) \quad (x \in \mathbb{R}^V),$$

with zero-weight edges omitted.

Proof. Discard zero-weight hyperedges and apply Theorem 1.1 to the seminorms in (17). Set $\mathcal{E}' = \text{supp}(w)$ and $c'_e = w_e c_e$. The common kernel of the N_e consists exactly of vectors that are constant on each connected component of the positive-weight 2-section. Its dimension is $\kappa(H)$, so the effective dimension is $n - \kappa(H)$. Evaluating at indicator vectors gives the cut inequalities. $\square$

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