Regular Distributions Are Not Closed Under Convolution

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Abstract

We show that Myerson regularity is not preserved by convolution, even for two independent identically distributed bounded nonnegative summands. For an absolutely continuous distribution F with density f, regularity means that the virtual value $x - (1 - F(x))/f(x)$ is nondecreasing at points where f and the survival probability are positive. Let $a = 1/50$, $c = 49/100$, $d = 51/100$, and $B = a/d^2 = 200/2601$. We define a common marginal law supported on $[0, 1]$ by the survival function

$$\overline{F}(x) = \begin{cases} \dfrac{a}{a+x}, & 0 \le x \le c, \\ B(1-x), & c \le x \le 1, \end{cases}$$

with $\overline{F}(x) = 1$ for $x < 0$ and $\overline{F}(x) = 0$ for $x > 1$. This marginal is regular, with virtual value $-a$ on $[0, c]$ and $2x - 1$ on $[c, 1)$. If S is the sum of two independent copies and Φ is the virtual value of S, then

$$\frac{d}{dt}\Phi(1+t)\bigg|_{t=0+} = \frac{31255005}{12510002} - \frac{6759999}{6255001}\log\frac{51}{2} < 0.$$

Thus Φ decreases immediately to the right of the interior point 1 of the support of S, and the distribution of S is not regular.

Note. This paper was fully AI generated by GPT 5.5 at xhigh reasoning effort using an internal harness, prompted by Richard Peng.

Contents

1 Introduction

Regularity is one of the basic shape assumptions in optimal auction theory. For an absolutely continuous distribution F with density f and survival function $\overline{F} = 1 - F$, the associated virtual value is

$$\varphi_F(x) = x - \frac{\overline{F}(x)}{f(x)}$$

at points where both $f(x)$ and $\overline{F}(x)$ are positive. The distribution is regular, in the sense of Myerson, if φ_F is nondecreasing on the part of the support on which it is defined [Mye81]. This monotonicity is what lets optimal-auction rules be expressed without ironing; the same virtual-value calculus also appears in the classical marginal-revenue interpretation of optimal auctions and monopoly pricing [RS81, BR89].

It is natural to ask whether regularity is preserved by aggregation. Given independent nonnegative values $X_1, \ldots, X_m$, each with a regular marginal, does the sum

$$S = X_1 + \cdots + X_m$$

again have a regular distribution? Equivalently, is the class of Myerson-regular distributions closed under convolution? This is not a formal consequence of the marginal definition: the virtual value of the sum depends on the ratio of the convolution survival function to the convolution density, and that ratio need not inherit monotonicity from the corresponding marginal ratios.

Positive closure results are known for stronger or different distributional hypotheses. The closest benchmark for the same aggregation operation is the monotone-hazard-rate, or IFR/MHR, condition: if the hazard rate $h_F = f/\overline{F}$ is nondecreasing, then $1/h_F$ is nonincreasing and $x - 1/h_F(x)$ is nondecreasing, so MHR implies Myerson regularity. Since IFR/MHR is a stronger assumption placed on the same convolution question, Barlow, Marshall, and Proschan's closure theorem for IFR/MHR distributions is the main positive restricted comparison [BMP63]. Log-concavity is another common positive regime: the Prékopa theorem gives stability under marginalization and convolution, and standard one-dimensional log-concavity facts imply stronger survival and hazard-rate properties in economic applications [Pré73, BB05]. Separately, regularity and its quantitative strengthenings have been studied in mechanism design under formulations such as generalized concavity, α-strong regularity, and related performance assumptions [Ewe13, CR14, CR15, SS19]. Revenue-management assumptions such as increasing generalized failure rate are nearby but different, since they involve monotonicity of $xh_F(x)$ and related pricing quantities rather than monotonicity of $x - 1/h_F(x)$ [Lar06, ZAF04, BM13]. These works provide important context and stronger or adjacent sufficient conditions, but they do not imply convolution closure for ordinary Myerson regularity.

The contribution of this paper is an explicit counterexample showing that the ordinary regularity assumption is too weak for such a closure theorem. The failure already occurs for the smallest nontrivial convolution, with two independent identically distributed summands supported on $[0, 1]$.

Theorem 1.1 (Explicit i.i.d. convolution counterexample)**.** *Let*

$$a = \frac{1}{50}, \qquad c = \frac{49}{100}, \qquad d = \frac{51}{100}, \qquad B = \frac{a}{d^2} = \frac{200}{2601}.$$

There exist independent identically distributed nonnegative continuously distributed random variables X_1, X_2, supported on $[0, 1]$, whose common survival function on $0 \le x \le 1$ is

$$\overline{F}(x) = \begin{cases} \dfrac{a}{a+x}, & 0 \le x \le c, \\ B(1-x), & c \le x \le 1 \end{cases}$$

with $\overline{F}(x) = 1$ for $x < 0$ and $\overline{F}(x) = 0$ for $x > 1$. The common distribution is regular. However, if $S = X_1 + X_2$ and Φ is the virtual value of the distribution of S, then

$$\frac{d}{dt}\Phi(1+t)\bigg|_{t=0+} = \frac{31255005}{12510002} - \frac{6759999}{6255001}\log\frac{51}{2} < 0.$$

Consequently the distribution of S is not regular.

The construction is intentionally elementary. The marginal survival function is chosen so that its inverse hazard ratio is $a + x$ on the first interval and $1 - x$ on the second interval. Thus the marginal virtual value is constant and then increasing, with no downward jump at the breakpoint. For the sum of two copies, we examine only the one-sided neighborhood $s = 1 + t$ of the interior point 1. In this range the convolution density and survival probability have closed forms, and differentiating their quotient gives the negative derivative displayed in Theorem 1.1.

The rest of the paper is organized as follows. Section 2 records the notation and basic convolution identities used throughout. Section 3 gives the short proof roadmap and explains why the chosen parameters make the computation local. Section 4 proves the regularity of the marginal, derives the local convolution formulas, and establishes the negative virtual-value slope for the sum.

2 Preliminaries

All random variables below are nonnegative and absolutely continuous. For a random variable X with distribution function F and density f, write

$$\overline{F}(x) = 1 - F(x)$$

for its survival function. We extend densities by zero outside their supports. The inverse hazard ratio and virtual value are

$$r_F(x) = \frac{\overline{F}(x)}{f(x)}, \qquad \varphi_F(x) = x - r_F(x), \tag{1}$$

at points where $f(x) > 0$ and $\overline{F}(x) > 0$. A distribution is regular if φ_F is nondecreasing on this set. For the compactly supported piecewise smooth distributions used here, the right endpoint is excluded because the survival probability is zero, and matching one-sided values at a breakpoint are read as the value of the displayed formula there.

If X_1, X_2 are independent copies of X and $S = X_1 + X_2$, write H for the distribution function of S, g for its density, and $\overline{H} = 1 - H$ for its survival function. The only convolution identities used below are

$$g(s) = \int_{\mathbb{R}} f(x)f(s-x)\,dx, \qquad \overline{H}(s) = \int_{\mathbb{R}} f(x)\overline{F}(s-x)\,dx. \tag{2}$$

The virtual value of the sum is

$$\Phi(s) = s - \frac{\overline{H}(s)}{g(s)} \tag{3}$$

where $g(s) > 0$ and $\overline{H}(s) > 0$. If Φ has a negative right derivative at an interior support point, then Φ is not nondecreasing. All logarithms are natural.

3 Overview

The proof gives a direct i.i.d. counterexample. Rather than trying to control regularity under an arbitrary convolution, we choose a marginal whose inverse hazard ratio is completely explicit on two intervals. The constants

$$a = \frac{1}{50}, \qquad c = \frac{49}{100}, \qquad d = 1 - c = \frac{51}{100}, \qquad B = \frac{a}{d^2}$$

are chosen so that $a + c = d$. This makes the survival function

$$\overline{F}(x) = \begin{cases} \dfrac{a}{a+x}, & 0 \le x \le c, \\ B(1-x), & c \le x \le 1 \end{cases}$$

fit together continuously, and in fact makes the density pieces agree at the breakpoint. The motivation for this normalization is that the inverse hazard ratio is then $a + x$ on the first piece and $1 - x$ on the second piece. Thus the marginal virtual value is $-a$ followed by $2x - 1$, and these two values agree at $x = c$. This proves the marginal regularity in Lemma 4.1 without any hidden smoothing argument.

The nontrivial part is that convolution regularity depends on the ratio $\overline{H}(s)/g(s)$ for the sum, not on the two marginal virtual values separately. The proof therefore looks only at a one-sided neighborhood of the interior point $s = 1$ in the support of $S = X_1 + X_2$. Writing $s = 1 + t$, the choice $c < 1/2$ ensures that for $0 \le t \le c$ the convolution integrals have no low-low region. They split only into low-high, high-high, and high-low pieces, which leaves elementary formulas for

$$G(t) = g(1+t) = 2B\left(\frac{a}{a+t} - \frac{a}{d}\right) + B^2(a+t)$$

and

$$T(t) = \overline{H}(1+t) = Ba\left(2\log\frac{d}{a+t} + \frac{a+t}{d} - 1\right) + \frac{B^2}{2}\left(d^2 - (c-t)^2\right).$$

Lemma 4.2 derives these identities directly from the convolution formulas for the density and survival probability.

The final step is to evaluate the virtual value of the sum,

$$\Phi(s) = s - \frac{\overline{H}(s)}{g(s)}, \qquad \Psi(t) = \Phi(1+t) = 1 + t - \frac{T(t)}{G(t)}.$$

Differentiating the displayed local formulas at $t = 0$ gives

$$\Psi'(0+) = \frac{31255005}{12510002} - \frac{6759999}{6255001}\log\frac{51}{2}.$$

The proof then supplies an elementary sign certificate: $\log(51/2) > 3$, so the last quantity is negative. Hence Φ decreases immediately to the right of the interior point 1, which is exactly the obstruction to regularity needed in Lemma 4.3. Combining this with the regularity of the common marginal proves Theorem 1.1: regularity is not closed under convolution, even for two i.i.d. nonnegative continuous summands.

4 Proof of the Main Theorem

Set

$$a = \frac{1}{50}, \qquad c = \frac{49}{100}, \qquad d = 1 - c = \frac{51}{100}, \qquad B = \frac{a}{d^2} = \frac{200}{2601}.$$

The identity $a + c = d$ will make the two pieces of the survival function meet continuously at c.

The first lemma verifies that the prescribed marginal is a regular absolutely continuous distribution. The point of the construction is that the inverse hazard ratio has two elementary pieces whose associated virtual values join without a downward jump.

Lemma 4.1 (Regular marginal construction)**.** *Let F be the distribution on $[0, 1]$ with survival function*

$$\overline{F}(x) = \begin{cases} \dfrac{a}{a+x}, & 0 \le x \le c, \\ B(1-x), & c \le x \le 1 \end{cases} \tag{4}$$

with $\overline{F}(x) = 1$ for $x < 0$ and $\overline{F}(x) = 0$ for $x > 1$. Then F is absolutely continuous with density

$$f(x) = \begin{cases} \dfrac{a}{(a+x)^2}, & 0 \le x < c, \\ B, & c < x < 1, \end{cases} \tag{5}$$

and F is regular.

Proof. Since $a + c = d$, the two displayed formulas for $\overline{F}$ agree at c:

$$\frac{a}{a+c} = \frac{a}{d} = Bd.$$

Also $\overline{F}(0) = 1$ and $\overline{F}(1) = 0$. The function $\overline{F}$ is nonincreasing and absolutely continuous on $[0, 1]$, so it defines an absolutely continuous distribution with density $f(x) = -\overline{F}'(x)$ at every non-breakpoint point. This gives Equation (5). The two density pieces also agree at the breakpoint because

$$\frac{a}{(a+c)^2} = \frac{a}{d^2} = B.$$

For $0 \le x < c$,

$$\frac{\overline{F}(x)}{f(x)} = \frac{a/(a+x)}{a/(a+x)^2} = a + x,$$

so by Equation (1) the virtual value is

$$x - \frac{\overline{F}(x)}{f(x)} = -a.$$

For $c < x < 1$,

$$\frac{\overline{F}(x)}{f(x)} = \frac{B(1-x)}{B} = 1 - x,$$

so by Equation (1) the virtual value is

$$x - \frac{\overline{F}(x)}{f(x)} = 2x - 1.$$

At the joining point, $2c - 1 = -1/50 = -a$. Hence the virtual value is constant on $[0, c]$ and increasing on $[c, 1)$, and F is regular. $\square$

It remains to show that regularity fails after adding two independent copies. Only a one-sided neighborhood of the interior point 1 is needed. In that neighborhood the low-low integration region is absent because $1 > 2c$, leaving only low-high, high-high, and high-low contributions.

Lemma 4.2 (Local convolution formula)**.** *Let X_1, X_2 be independent with the distribution in Lemma 4.1, and let $S = X_1 + X_2$. Let g be the density of S and let $\overline{H}(s) = \Pr[S \geq s]$. For every $0 \leq t \leq c$, with $s = 1 + t$,*

$$g(1+t) = 2B\left(\frac{a}{a+t} - \frac{a}{d}\right) + B^2(a+t) \tag{6}$$

and

$$\overline{H}(1+t) = Ba\left(2\log\frac{d}{a+t} + \frac{a+t}{d} - 1\right) + \frac{B^2}{2}\left(d^2 - (c-t)^2\right). \tag{7}$$

Proof. For $s = 1 + t$ with $0 \leq t \leq c$, Equation (2) gives

$$g(s) = \int_{s-1}^{1} f(x)f(s-x)\,dx.$$

Because $s > 2c$, there is no interval on which both $x \leq c$ and $s - x \leq c$. Splitting at $x = c$ and $x = s - c = d + t$ gives

$$g(s) = \int_t^c \frac{a}{(a+x)^2} B\,dx + \int_c^{d+t} B^2\,dx + \int_{d+t}^1 B\frac{a}{(a+s-x)^2}\,dx.$$

The first and third integrals are equal by the substitution $y = s - x$. Since $s - 2c = a + t$ and $a + c = d$,

$$g(1+t) = 2B\int_t^c \frac{a}{(a+x)^2}\,dx + B^2(a+t) = 2B\left(\frac{a}{a+t} - \frac{a}{d}\right) + B^2(a+t).$$

This proves Equation (6).

For the survival probability, Equation (2) gives

$$\overline{H}(s) = \int_{s-1}^{1} f(x)\overline{F}(s-x)\,dx,$$

which holds for $1 \leq s \leq 2$. The same split gives

$$\overline{H}(s) = \int_t^c \frac{a}{(a+x)^2} B(1-s+x)\,dx + \int_c^{d+t} B^2(1-s+x)\,dx + \int_{d+t}^1 B\frac{a}{a+s-x}\,dx.$$

Since $s = 1 + t$, the first integral is

$$B\int_t^c \frac{a(x-t)}{(a+x)^2}\,dx = Ba\left(\log\frac{d}{a+t} + \frac{a+t}{d} - 1\right).$$

The middle integral is

$$B^2\int_c^{d+t}(x-t)\,dx = \frac{B^2}{2}\left(d^2 - (c-t)^2\right),$$

and the last integral is

$$Ba\int_{d+t}^1 \frac{dx}{a+1+t-x} = Ba\log\frac{d}{a+t}.$$

Adding these three identities proves Equation (7). $\square$

The final lemma differentiates the local formula. The derivative is evaluated at an interior point of the support of S, so a negative right derivative forces a local decrease of the convolution virtual value.

Lemma 4.3 (Negative convolution virtual slope)**.** *For $S = X_1 + X_2$ as in Lemma 4.2, the virtual value*

$$\Phi(s) = s - \frac{\overline{H}(s)}{g(s)} \tag{8}$$

is not nondecreasing on the support of S.

Proof. Write $\Psi(t) = \Phi(1+t)$ for $0 \le t \le c$. By Equations (6) and (7), put

$$\begin{aligned} G(t) &= g(1+t) = 2B\left(\frac{a}{a+t} - \frac{a}{d}\right) + B^2(a+t), \\ T(t) &= \overline{H}(1+t) = Ba\left(2\log\frac{d}{a+t} + \frac{a+t}{d} - 1\right) + \frac{B^2}{2}\left(d^2 - (c-t)^2\right). \end{aligned} \tag{9}$$

Then $\Psi(t) = 1 + t - T(t)/G(t)$. At $t = 0$,

$$G(0) = \frac{1000400}{6765201}, \qquad G'(0) = -\frac{51980000}{6765201},$$

and

$$T(0) = \frac{8}{2601}\log\frac{51}{2} - \frac{9596}{6765201}, \qquad T'(0) = -\frac{1000400}{6765201}.$$

Differentiating $\Psi(t) = 1 + t - T(t)/G(t)$ from the right at 0 gives

$$\Psi'(0+) = 1 - \frac{T'(0)G(0) - T(0)G'(0)}{G(0)^2} = \frac{31255005}{12510002} - \frac{6759999}{6255001}\log\frac{51}{2}. \tag{10}$$

We next give a numerical sign certificate using only elementary estimates. Since

$$e = 1 + 1 + \frac{1}{2} + \sum_{n\ge 3}\frac{1}{n!} \le \frac{5}{2} + \sum_{k\ge 0}\frac{1}{6\cdot 4^k} = \frac{49}{18} < \frac{11}{4},$$

where $n! \ge 6 \cdot 4^{n-3}$ for $n \ge 3$, we have $e^3 < 51/2$ and hence $\log(51/2) > 3$. Therefore

$$\Psi'(0+) < \frac{31255005}{12510002} - 3\frac{6759999}{6255001} = -\frac{9304989}{12510002} < 0.$$

By Equation (10) and the definition of the right derivative, there is some sufficiently small $\epsilon > 0$ such that

$$\Phi(1+\epsilon) = \Psi(\epsilon) < \Psi(0) = \Phi(1).$$

Thus Φ is not nondecreasing on the support of S. $\square$

Proof of Theorem 1.1. Let X_1 and X_2 be independent random variables with the common distribution constructed in Lemma 4.1. That lemma shows that the common distribution is regular. The proof of Lemma 4.3 computes the right derivative of the virtual value of $S = X_1 + X_2$ at 1 as the quantity in Equation (10) and proves that it is negative. Therefore the virtual value of S decreases immediately to the right of 1, and the distribution of S is not regular. This proves that regularity is not closed under convolution, even for two i.i.d. nonnegative continuously distributed random variables. $\square$

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