Faster Algorithms for Unit Maximum Flow

Yang P. Liu and Aaron Sidford

arXiv : 1910.14276, arxiv : 2003.08929

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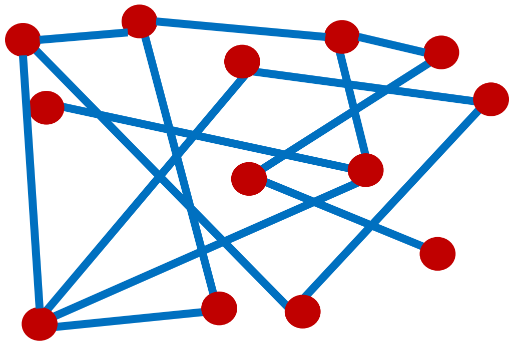
Talk Outline

Recent Advances in Flow Problems

Part 1

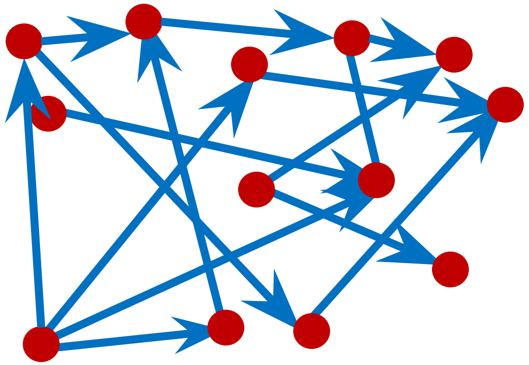


Energy Maximization of Electric Flows

Part 2

Beyond Electric Flows

Part 3



Putting it All Together: Full Algorithm

Part 4

The Maximum Flow Problem

Graph $G = (V, E)$

- n vertices V
- m edges E

Capacities

- $u \in \{1, \ldots, U\}^E$

Terminals

- Source $s \in V$
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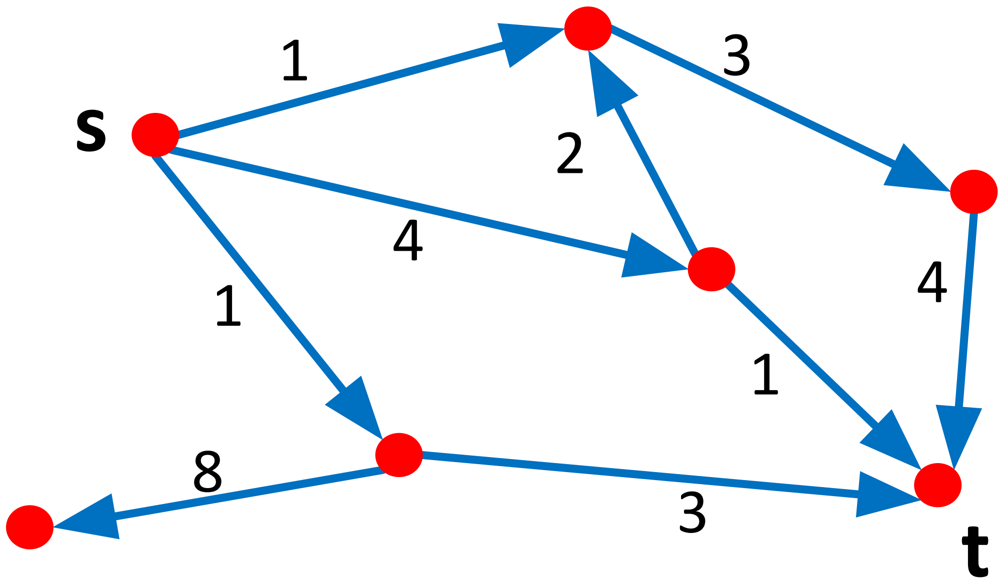

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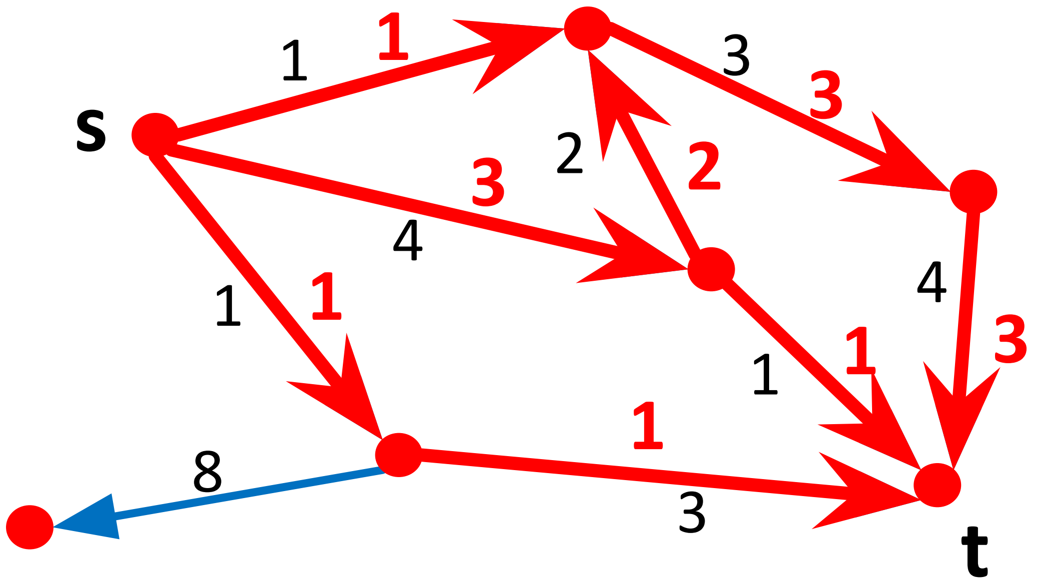

Goal
compute maximum $s \to t$ flow

Flow
$f \in \mathbb{R}^E$ where $f_e =$ amount of flow on edge e

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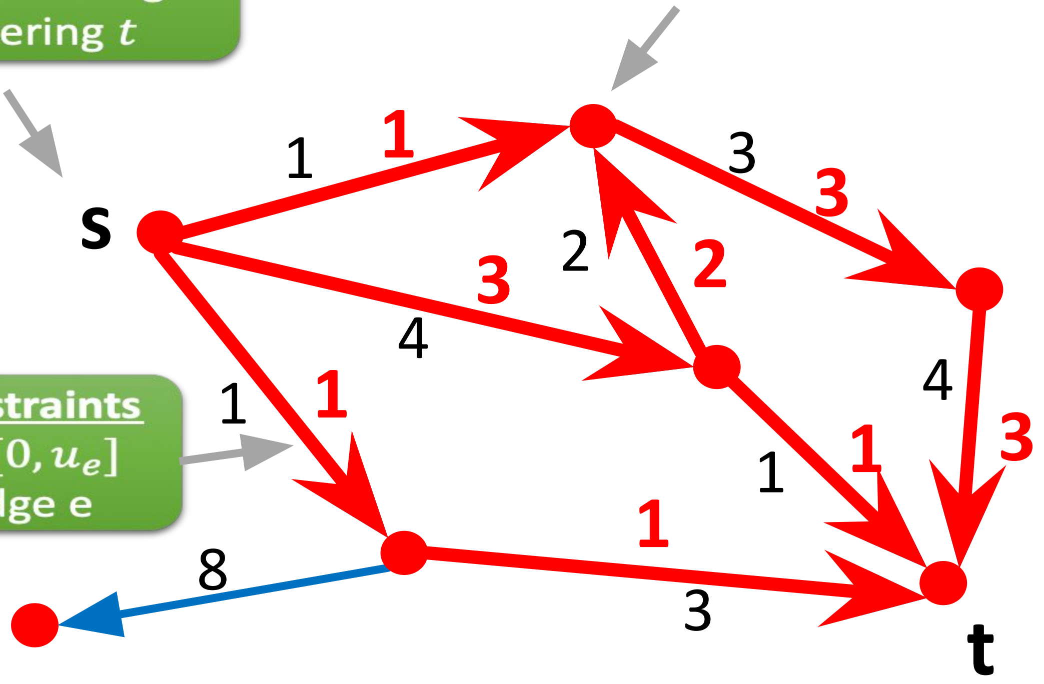
Value of Flow
total flow leaving s
or entering t

$s \rightarrow t$ Flow
flow in = flow out
for all $v \notin \{s, t\}$

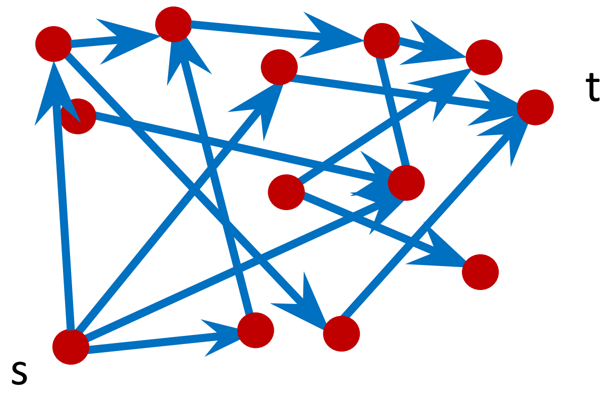


Capacity Constraints
flow on $e \in [0, u_e]$
for every edge e

Flow
$f \in \mathbb{R}^E$ where $f_e =$
amount of flow on edge e

Goal
compute maximum $s \rightarrow t$ flow

Why?



Fundamental

- Well studied with decades of extensive research
- Historically improvements yielded general techniques.

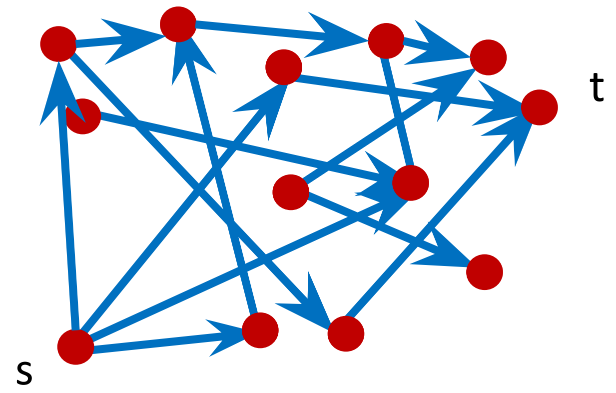
Applications

- Minimum s-t cut, bipartite matching, scheduling
- Subroutine for many problems: transportation, partitioning, clustering, etc.
- Captures difficulty of broader problems multicommodity flow, minimum cost flow, optimal transport, etc.

Simple “difficult” structured optimization problem

- Barrier for both continuous and discrete methods
- Captures core issues in algorithmic graph theory and “structured optimization”

Why?



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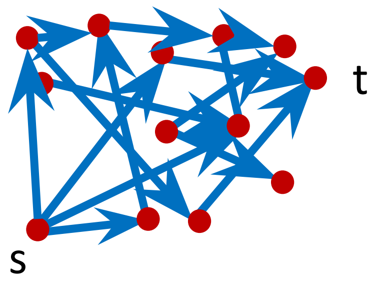
Improvements yield broad tools.

Proving ground for optimization techniques

Running Times



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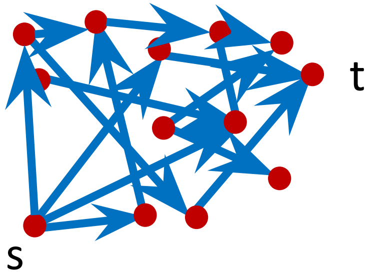

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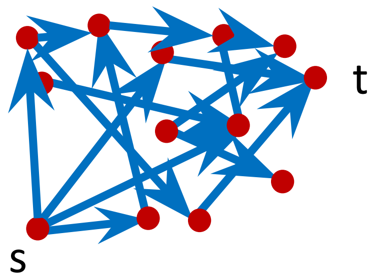

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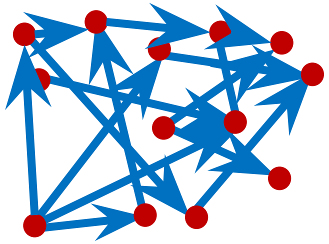
Open Question:
Can we achieve almost linear $m^{1+o(1)}$ time?

Running Times



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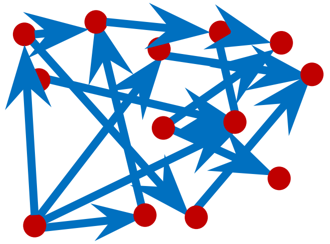

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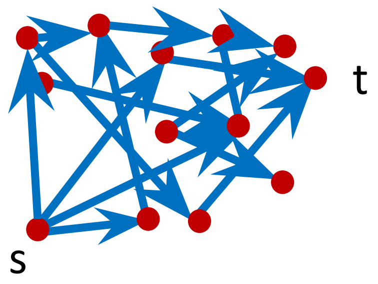

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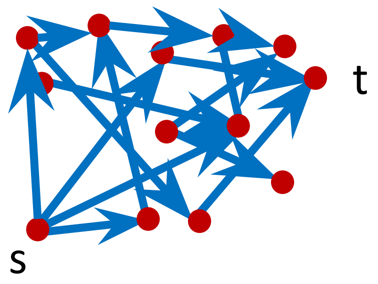

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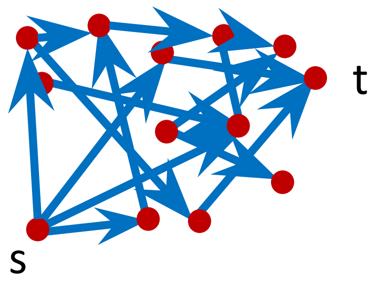
- Bipartite matching is $U = 1$ case
- Same runtime for minimum s-t cut

Running Times



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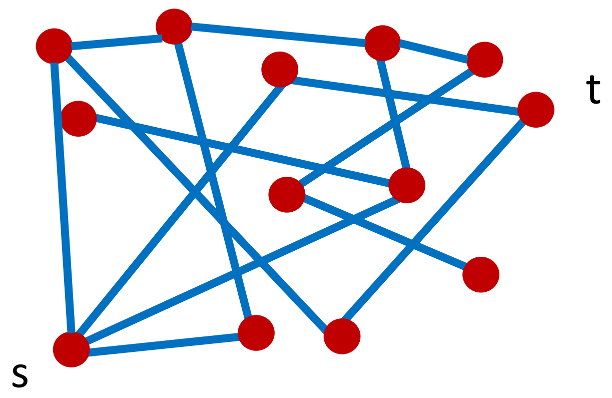
[AMV20] Mincost flows in time $m^{4/3+o(1)}\log C$

[BLNPSSSW20] Bipartite matching and transhipment in $O((m+n^{1.5})\log^2 W)$

Undirected Flow Problems

Natural family of problems in combinatorial optimization.

- Graph $G = (V, E)$
- Vertices $s, t \in V$



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What should we minimize?

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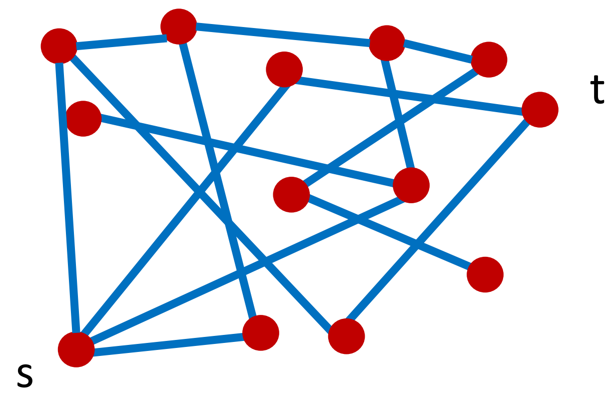

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Send 1 unit of flow, $f \in \mathbb{R}^E$, between s and t in the "best" way possible.

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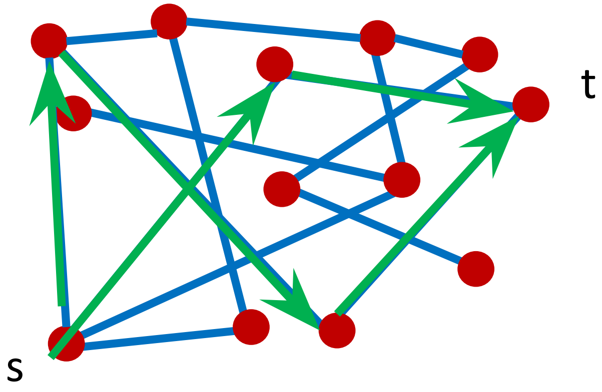

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Maximum Flow

$\tilde{O}(|E|\sqrt{|V|})$, $\tilde{O}(|E|^{10/7})$
[LS14] [M13]

Congestion
$\max_{e \in E} |f_e|$

$\|f\|_\infty$

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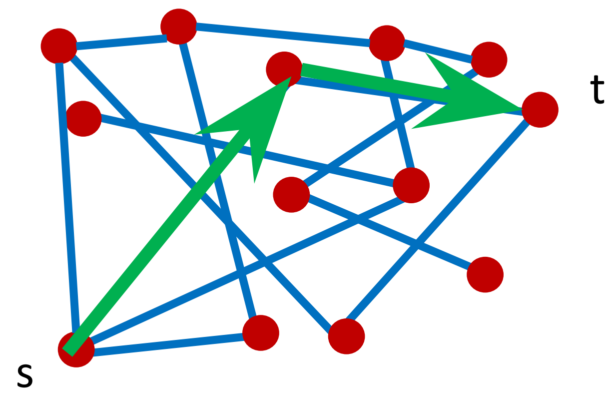

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Shortest Path

$\tilde{O}(|E|)$

Length

$\sum_{e \in E} |f_e|$

$\|f\|_1$

Maximum Flow

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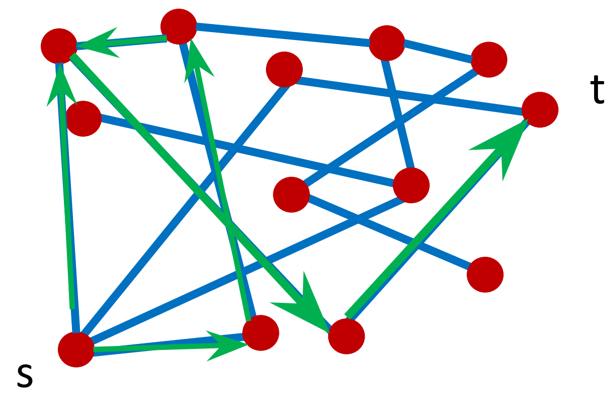

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What should we minimize?

Shortest Path $\tilde{O}(\lvert E \rvert)$	**Length** $\sum_{e \in E} \lvert f_e \rvert$	$\lVert f \rVert_1$
Electric Flow Laplacian System Solving $\tilde{O}(\lvert E \rvert)$ [ST04]	**Energy** $\sum_{e \in E} \lvert f_e \rvert^2$	$\lVert f \rVert_2$
Maximum Flow $\tilde{O}(\lvert E \rvert \sqrt{\lvert V \rvert})$, $\tilde{O}(\lvert E \rvert^{10/7})$ [LS14] [M13]	**Congestion** $\max_{e \in E} \lvert f_e \rvert$	$\lVert f \rVert_\infty$

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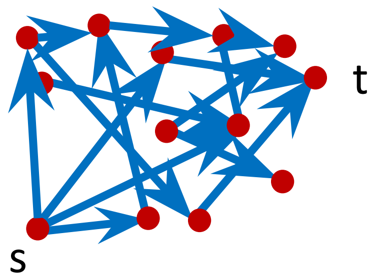

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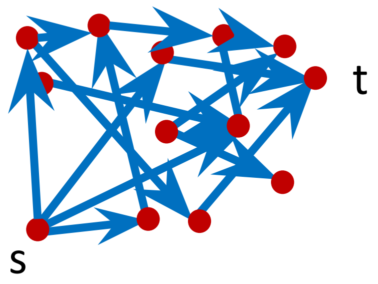

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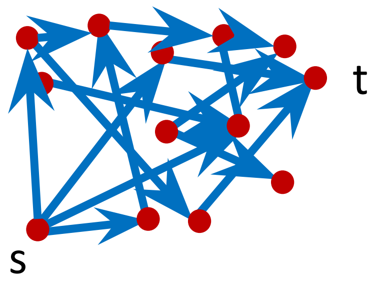

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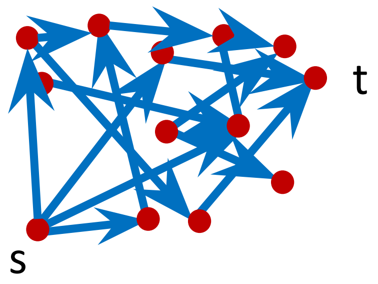

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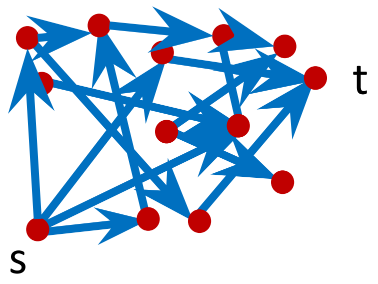

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Iterate on something stronger?

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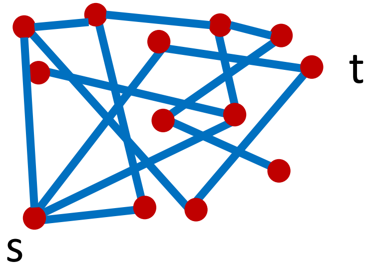
Undirected Graphs



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feasible $s \to t$ flow of value $(1-\epsilon)OPT$

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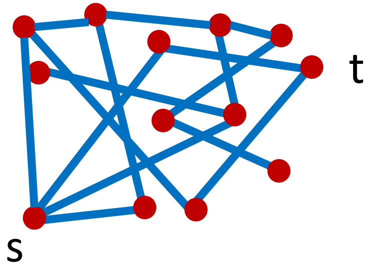

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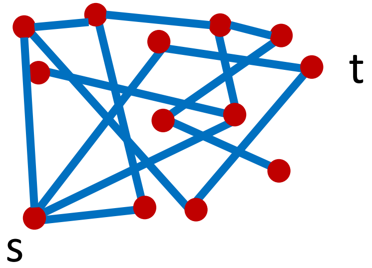

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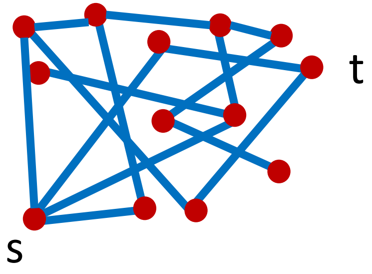

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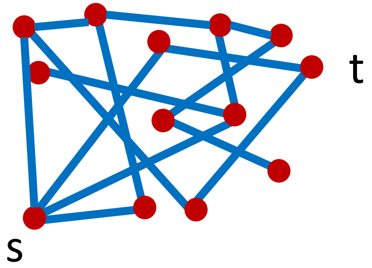
Undirected Graphs



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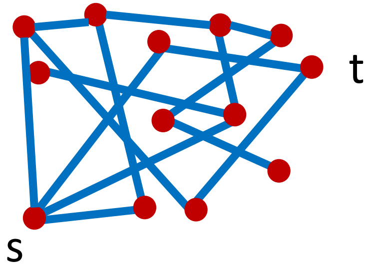

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Idea
Combine / apply these primitives in IPMs!

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Stronger primitives?

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Directed Laplacians

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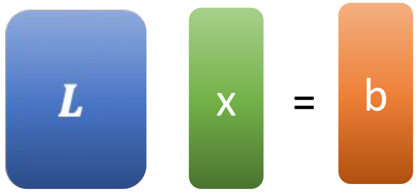
$\boldsymbol{L} = \boldsymbol{D}_{\text{out}}(G) - \boldsymbol{A}(G)$



- Directed, asymmetric variant of electric flow and Laplacians systems.
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- Can solve in nearly linear time!
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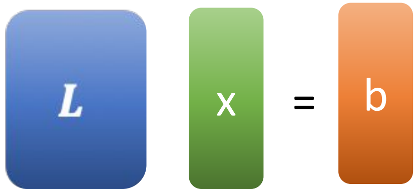

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ℓ_p-Flows

$$\min_{\substack{unit\ s\text{-}t \\ flow\ f \in \mathbb{R}^E}} \|f\|_{r,p}^p = \sum_{e \in E} r_e |f_e|^p$$

- [BCLL18, AKPS19]
- Faster algorithms using electric flow ($p = 2$ case)
- Input sparsity runtimes!

Stronger primitives?

Don't know how to use for directed maximum flow ☹

Don't know how to use to speed up IPMs ☹

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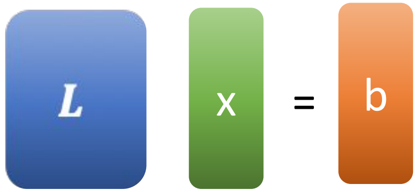

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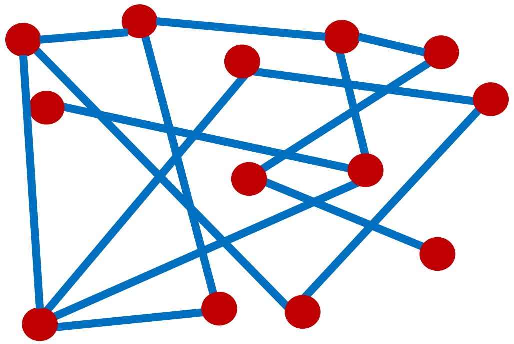
Maxflow-like Potential
Unweighted, high-power

Talk Outline

Recent Advances in Flow Problems

Part 1



Energy Maximization of Electric Flows

Part 2

Beyond Electric Flows

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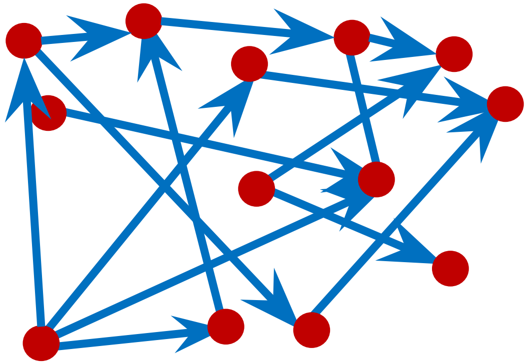

Putting it All Together: Full Algorithm

Part 4

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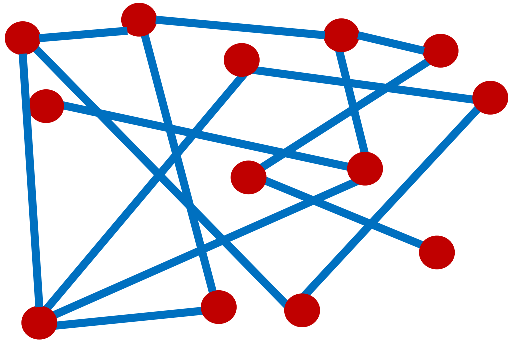

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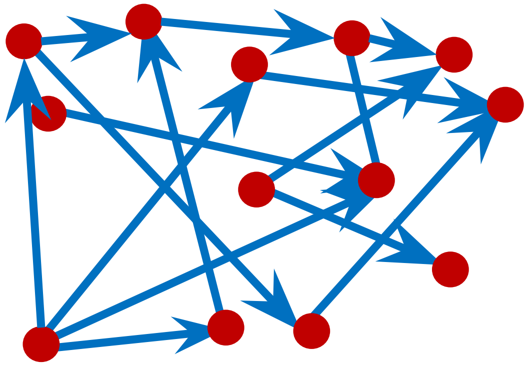

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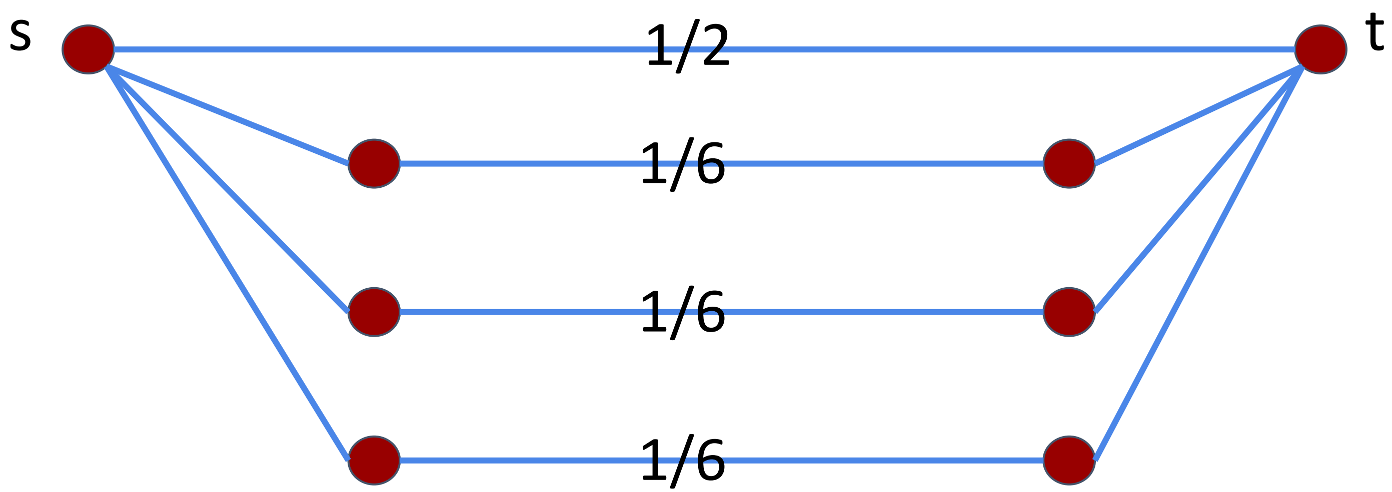

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** Not exactly the framework but close.*

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Assume undirected graph

Algorithm state:

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- **Goal:** Change weights to allow for larger progress steps (greater than $m^{-1/2}$)
- **Invariant:** Need to maintain $\sum_{e \in E} (w_e^+ + w_e^-) \leq O(m)$

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How to improve?

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- **Our approach:** solve weight budgeted energy maximization as its own optimization problem!

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Approximate solve: change $\infty \to p = \sqrt{\log m}$ and solve using smoothed ℓ_2-ℓ_p norm flow result of **[KPSW19].**

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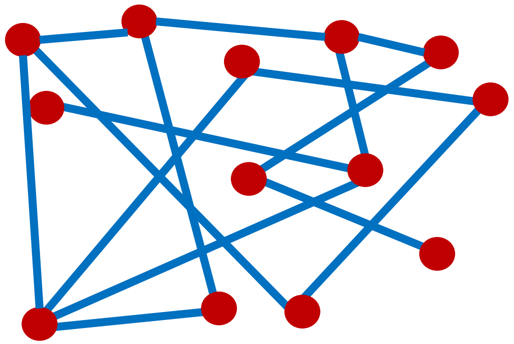
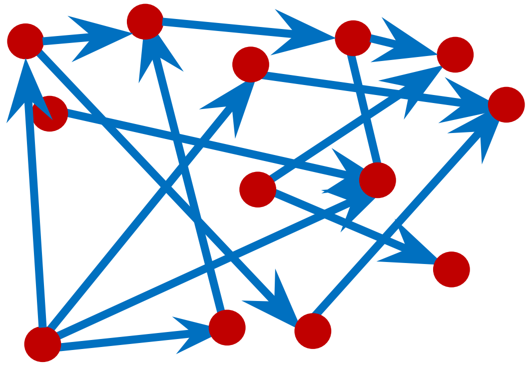


Energy Maximization of Electric Flows

Part 2

Beyond Electric Flows

Part 3



Putting it All Together: Full Algorithm

Part 4

Talk Outline

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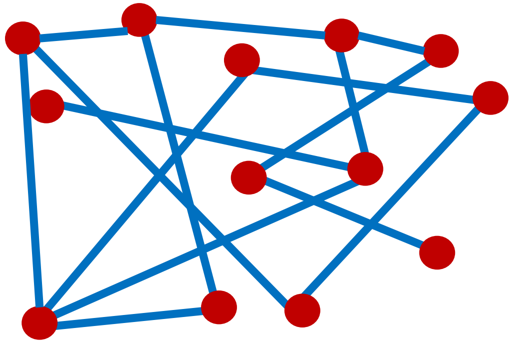

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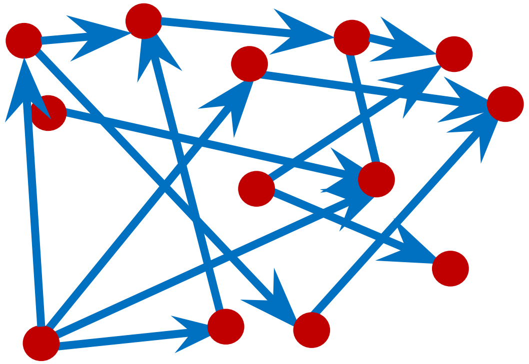

Putting it All Together: Full Algorithm

Part 4

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Approximately an electric flow!

Error of Electric Flow Approximation

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congestion

residual capacities

Congestion, Progress, and Correction

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Solution: Don't augment via electric flows, i.e. don't force $\hat{f}$ to be an electric flow!

Maintaining Centrality via Divergences

Goal: Augment by a flow $\hat{f}$ that doesn't cause centrality error via approximations, like the electric flow did.

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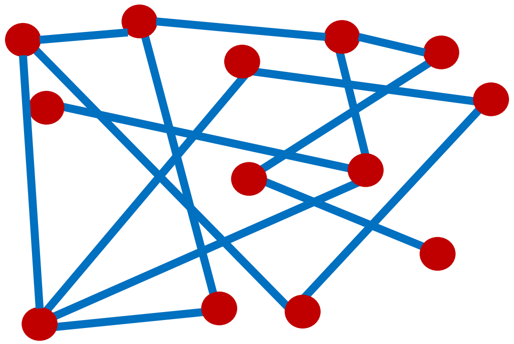
Intuition: Divergence is 2nd order, approximated by electric energy!

Talk Outline

Recent Advances in Flow Problems

Part 1

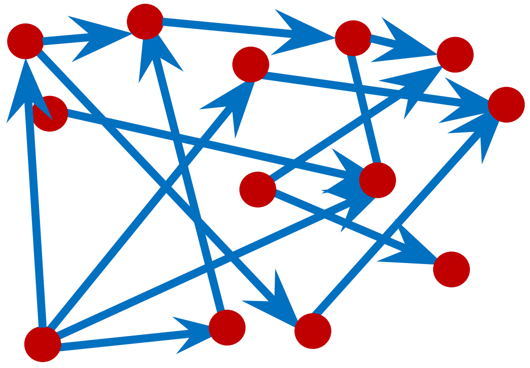


Energy Maximization of Electric Flows

Part 2

Beyond Electric Flows

Part 3



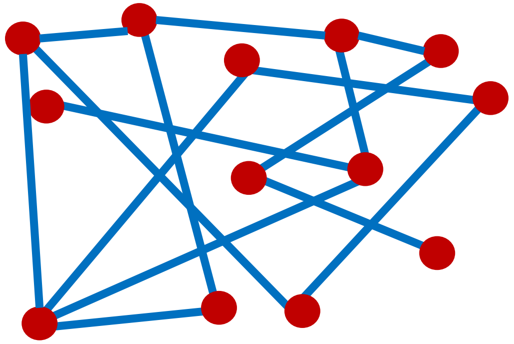
Putting it All Together: Full Algorithm

Part 4

Talk Outline

Recent Advances in Flow Problems

Part 1



Energy Minimization of Electrical Flows

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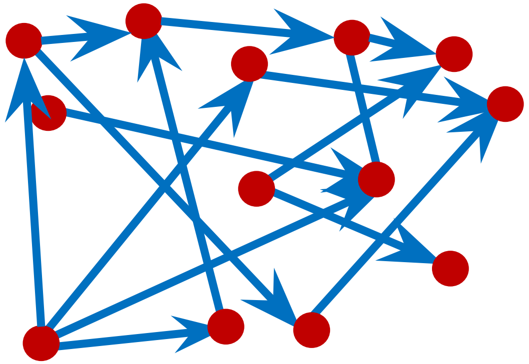

Putting it All Together: Full Algorithm

Part 4

Full Algorithm for $m^{4/3+o(1)}$ Time Maxflow

Step 1: Precondition the graph G.

Step 2: For $m^{1/3+o(1)}$ steps do

a. Perform energy maximization on the *divergence* objective.

b. Change weights accordingly, and augment flow.

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Weight change bound: Trade off size of progress steps, amount of flow left, and amount of weight change.

Energy Maximization in Almost Linear Time

The Energy Maximization Problem

$$\max_{\|Cr'\|_1 \le W} \text{energy}_{r+r'}(f)$$

$$= \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + W \left\|C^{-1/2} f\right\|_\infty^2$$

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For $p = \log^c n$ with $c \in (0,2/3)$ in can solve in almost linear time:

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Using *iterative refinement* [AKPS19, KPSW19], can solve *divergence minimization* problem using $m^{o(1)}$ instances of ℓ_2-ℓ_p norm flow

Parameter Tradeoffs: $m^{4/3}$ vs $m^{11/8}$

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c is a small constant -- we are aiming for $m^{3/2-c}$ runtime.

Lemma: If there are F units of residual flow, and we want to route δF units, then the congestion vector satisfies $\|\rho\|_2 \leq \delta\sqrt{m}$.

Note: For $\delta = m^{-1/2}$, this recovers standard IPMs. We want $\delta = m^{-1/2+c}$.

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For W = O(m^{4c}) we can guarantee $\|\rho\|_\infty \leq 1/10$: sufficient for divergence-based method.

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Total change over $m^{1/2-c}$ IPM steps = $m^{1/2-c} \times W = m^{1/2+3c} \leq m$ for $c = 1/6$.

Runtime = $m^{3/2-c} = m^{4/3}$.

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Choices: δ = $m^{-1/2+c}$ and W = O(m^{6c}) gives $\|\rho\|_2 \leq m^c$ and $\|\rho\|_\infty \leq m^{-c}/100.$

Therefore, we have $\|\rho\|_4 \leq (\|\rho\|_2\|\rho\|_\infty)^{1/2} \leq 1/10.$

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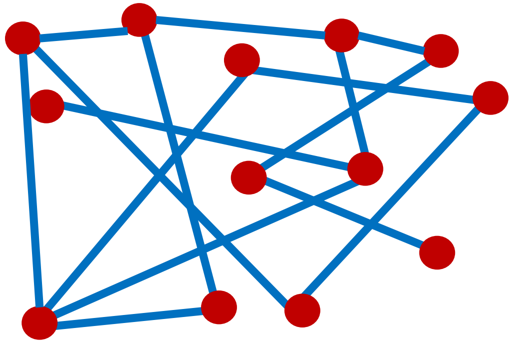
Total weight increase over $m^{1/2-c}$ steps is $m^{1/2-c}$ x W <= $m^{1/2+5c}$ <= m for c = 1/10. Gives runtime $m^{3/2-c}$ = $m^{7/5}$ = $m^{1.4}$. Larger than $m^{11/8}$ = $m^{1.375}$.

[LS19] need additional *weight reduction* tricks to get $m^{11/8}$.

Talk Outline

Recent Advances in Flow Problems

Part 1



Energy Maximization of Electrical Flows

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Beyond Electrical Flows

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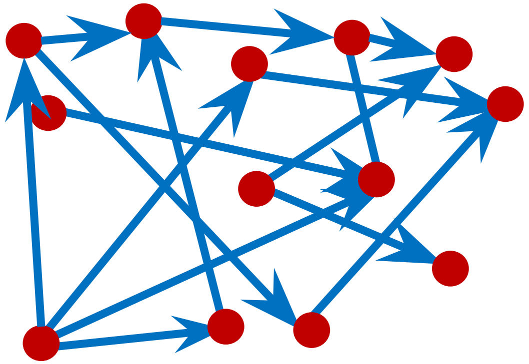

Putting it All Together: Full Algorithm

Part 4

Talk Outline

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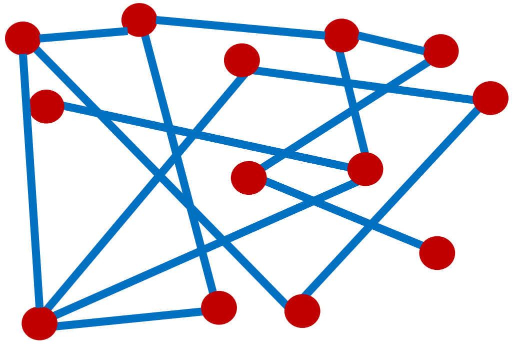

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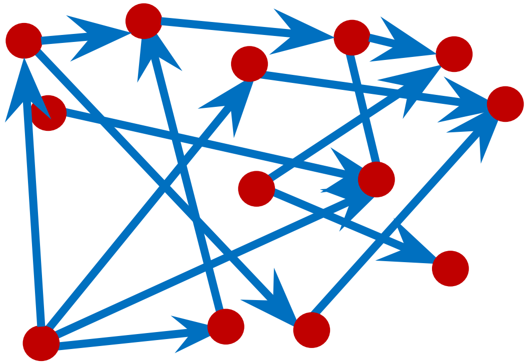

Putting it l Toge er: Full
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Part 4

Future Directions / Open Problems

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Better Primitives?

- Smoothed lp flows for weighted graphs

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- Can we go beyond IPMs?
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Approximation algorithms?

- **Generalization:** For any $\varepsilon > 0$, can compute εmU-additive approximate maxflow in time $m^{1+o(1)}/\varepsilon^{1/2}$.
- ε-approximate maxflow in $m^{1+o(1)}/\varepsilon^{1/2}$ time?

Faster Algorithms for Unit Maximum Flow

arXiv : 1910.14276

arxiv : 2003.08929

The End

Questions?

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Aaron Sidford

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