Faster Energy Maximization for Faster Maximum Flow

Yang P. Liu and Aaron Sidford

arXiv : 1910.14276

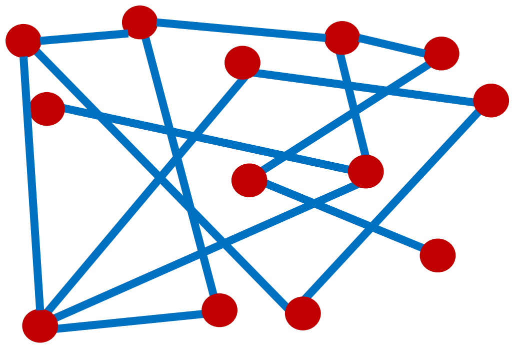
Contact Info:

- Email: yangpliu@stanford.edu
- Website: *yangpliu.github.io*

Talk Outline

Recent Advances in Flow Problems

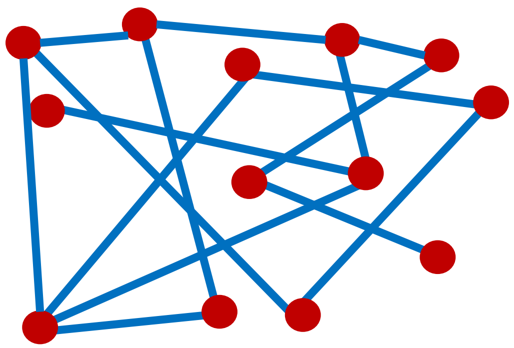
Part 1



Talk Outline

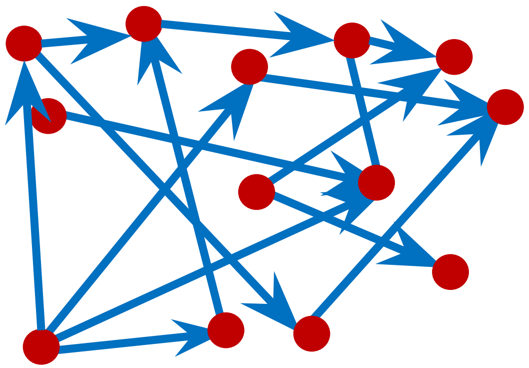
Recent Advances in Flow Problems

Part 1



[LS19] Energy Maximization and Maximum Flow

Part 2



The Maximum Flow Problem

Graph $G = (V, E)$

- n vertices V
- m edges E

Capacities

- $u \in \{1, \dots, U\}^E$

Terminals

- Source $s \in V$
- Terminal $t \in V$

The Maximum Flow Problem

Graph $G = (V, E)$

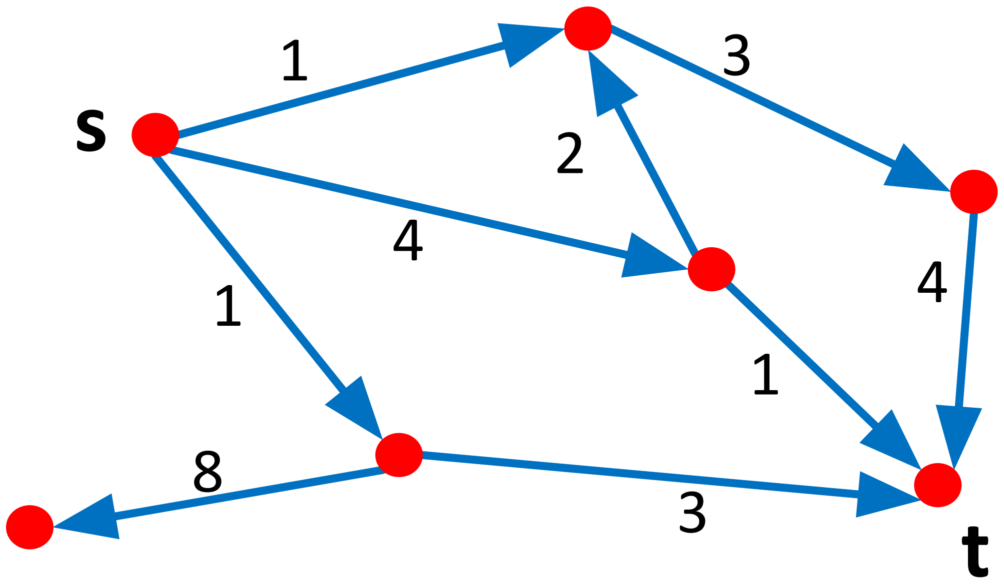
- n vertices V
- m edges E

Capacities

- $u \in \{1, \ldots, U\}^E$

Terminals

- Source $s \in V$
- Terminal $t \in V$



The Maximum Flow Problem

Graph $G = ({\color{red}V}, {\color{blue}E})$

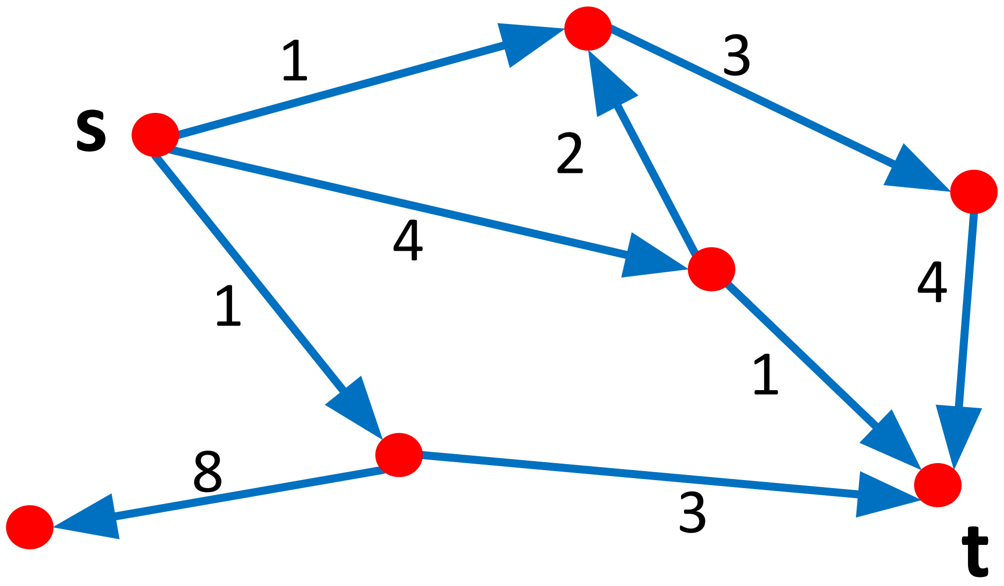
- n vertices V
- m edges E

Capacities

- $u \in \{1, \ldots, U\}^E$

Terminals

- Source $s \in V$
- Terminal $t \in V$



Flow
$f \in \mathbb{R}^E$ where $f_e =$ amount of flow on edge e

Goal
compute maximum $s \to t$ flow

The Maximum Flow Problem

Graph $G = (V, E)$

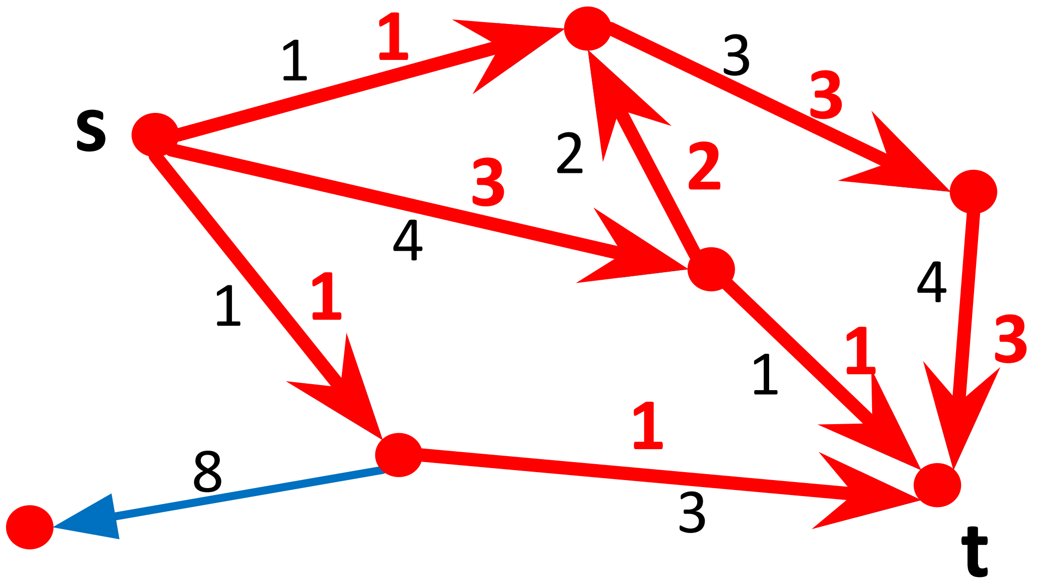
- n vertices V
- m edges E

Capacities

- $u \in \{1, \dots, U\}^E$

Terminals

- Source $s \in V$
- Terminal $t \in V$



Goal
compute maximum $s \to t$ flow

Flow
$f \in \mathbb{R}^E$ where $f_e =$ amount of flow on edge e

The Maximum Flow Problem

Graph $G = (V, E)$

- n vertices V
- m edges E

Capacities

- $u \in \{1, \dots, U\}^E$

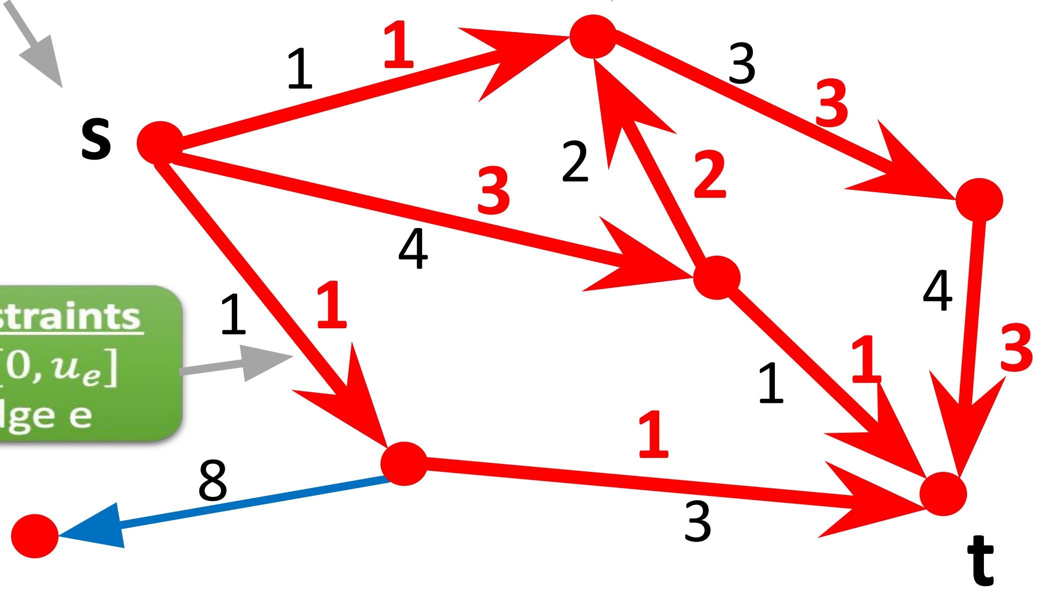
Terminals

- Source $s \in V$
- Terminal $t \in V$

Value of Flow
total flow leaving s
or entering t

$s \to t$ Flow
flow in = flow out
for all $v \notin \{s, t\}$

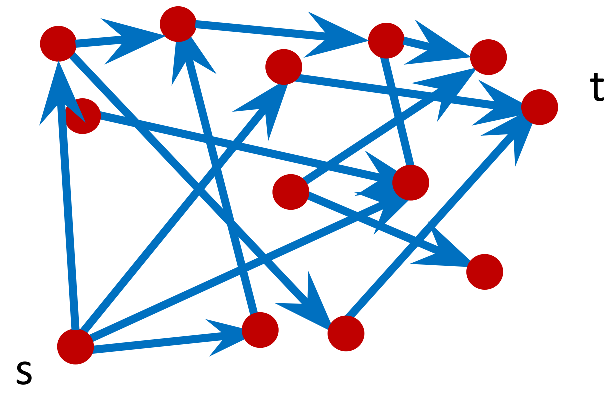
Capacity Constraints
flow on $e \in [0, u_e]$
for every edge e



Flow
$f \in \mathbb{R}^E$ where $f_e =$
amount of flow on edge e

Goal
compute maximum $s \to t$ flow

Why?



Fundamental

- Well studied with decades of extensive research
- Historically improvements yielded general techniques.

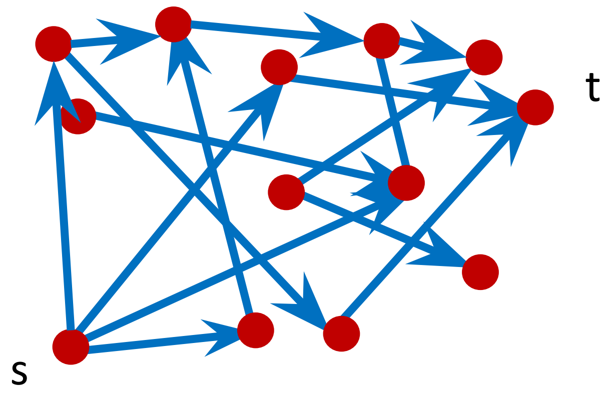
Applications

- Minimum s-t cut, bipartite matching, scheduling
- Subroutine for many problems: transportation, partitioning, clustering, etc.
- Captures difficulty of broader problems multicommodity flow, minimum cost flow, optimal transport, etc.

Simple “difficult” structured optimization problem

- Barrier for both continuous and discrete methods
- Captures core issues in algorithmic graph theory and “structured optimization”

Why?



Fundamental

- Well studied with decades of extensive research
- Historically improvements yielded general techniques.

Applications

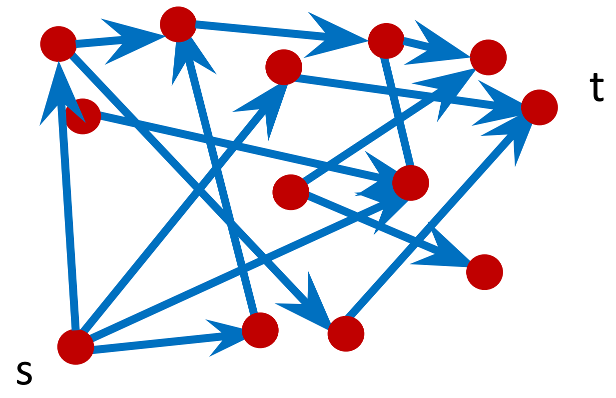
- Minimum s-t cut, bipartite matching, scheduling
- Subroutine for many problems: transportation, partitioning, clustering, etc.
- Captures difficulty of broader problems multicommodity flow, minimum cost flow, optimal transport, etc.

Improvements yield broad tools.

Simple "difficult" structured optimization problem

- Barrier for both continuous and discrete methods
- Captures core issues in algorithmic graph theory and "structured optimization"

Why?



Fundamental

- Well studied with decades of extensive research
- Historically improvements yielded general techniques.

Applications

- Minimum s-t cut, bipartite matching, scheduling
- Subroutine for many problems: transportation, partitioning, clustering, etc.
- Captures difficulty of broader problems multicommodity flow, minimum cost flow, optimal transport, etc.

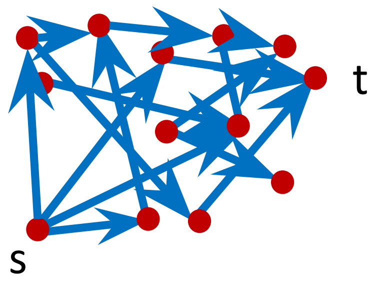
Simple "difficult" structured optimization problem

- Barrier for both continuous and discrete methods
- Captures core issues in algorithmic graph theory and "structured optimization"

Improvements yield broad tools.

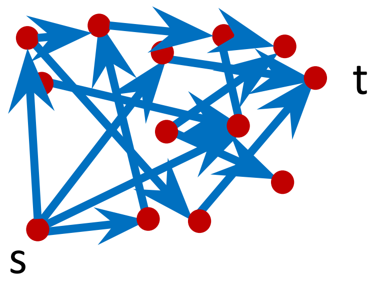
Proving ground for optimization techniques

Running Times



- *Graph* $G = (V, E)$, $|V| = n$, $|E| = m$
- *Capacities* $u \in \{1, \ldots, U\}^E$
- *Source* $s \in V$ *and terminal* $t \in V$

Running Times



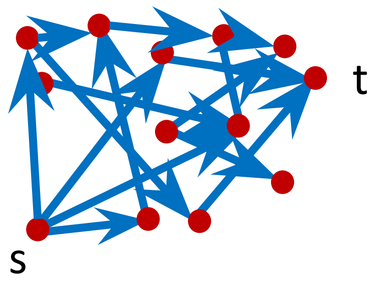
- *Graph* $G = (V, E)$, $|V| = n$, $|E| = m$
- *Capacities* $u \in \{1, \dots, U\}^E$
- *Source* $s \in V$ *and terminal* $t \in V$

Authors	Exact Maximum Flow Time	Unit Capacity Only?
[K73,ET75]	$\min\left(m^{3/2}, mn^{2/3}\right)$	Yes
[GR98]	$\tilde{O}\left(m^{3/2}\right)$	No
[M13]	$\tilde{O}(m^{10/7}U^{10/7})$	No
[LS14]	$\tilde{O}\left(m\sqrt{n}\right)$	No
[M16]	$\tilde{O}\left(m^{10/7}U^{1/7}\right)$	No

Use $\tilde{O}$ *to hide* $\log^{O(1)}(nU/\epsilon)$.

There is a $O(mn)$ *strongly polynomial time algorithm, i.e. no appearance of* U. *[O13]*

Running Times



- *Graph* $G = (V, E)$, $|V| = n$, $|E| = m$
- *Capacities* $u \in \{1, \ldots, U\}^E$
- *Source* $s \in V$ *and terminal* $t \in V$

Authors	Exact Maximum Flow Time	Unit Capacity Only?
[K73,ET75]	$\min\left(m^{3/2}, mn^{2/3}\right)$	Yes
[GR98]	$\tilde{O}\left(m^{3/2}\right)$	No
[M13]	$\tilde{O}(m^{10/7}U^{10/7})$	No
[LS14]	$\tilde{O}\left(m\sqrt{n}\right)$	No
[M16]	$\tilde{O}\left(m^{10/7}U^{1/7}\right)$	No

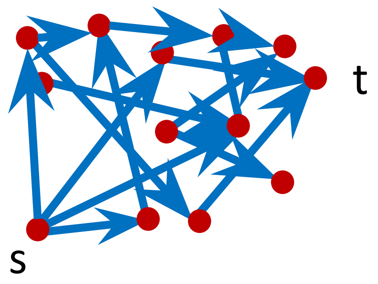
Use $\tilde{O}$ *to hide* $\log^{O(1)}(nU/\epsilon)$.

There is a $O(mn)$ *strongly polynomial time algorithm, i.e. no appearance of* U. *[O13]*

Our Result

$m^{11/8+o(1)}U^{1/4}$

Running Times



- *Graph* $G = (V, E)$, $|V| = n$, $|E| = m$
- *Capacities* $u \in \{1, \ldots, U\}^E$
- *Source* $s \in V$ *and terminal* $t \in V$

Authors	Exact Maximum Flow Time	Unit Capacity Only?
[K73,ET75]	$\min\left(m^{3/2}, mn^{2/3}\right)$	Yes
[GR98]	$\tilde{O}\left(m^{3/2}\right)$	No
[M13]	$\tilde{O}(m^{10/7}U^{10/7})$	No
[LS14]	$\tilde{O}\left(m\sqrt{n}\right)$	No
[M16]	$\tilde{O}\left(m^{10/7}U^{1/7}\right)$	No

Use $\tilde{O}$ *to hide* $\log^{O(1)}(nU/\epsilon)$.

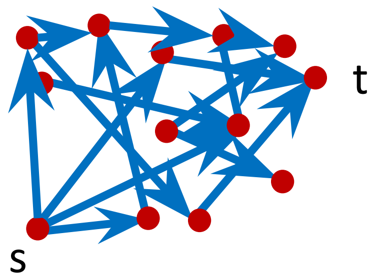
There is a $O(mn)$ *strongly polynomial time algorithm, i.e. no appearance of* U. *[O13]*

- $10/7 = 3/2 - 1/14$
- $11/8 = 3/2 - 1/8$
- $4/3 = 3/2 - 1/6$

Our Result

$m^{11/8+o(1)}U^{1/4}$

Running Times



- Graph $G = (V, E)$, $|V| = n$, $|E| = m$
- Capacities $u \in \{1, \ldots, U\}^E$
- Source $s \in V$ and terminal $t \in V$

Authors	Exact Maximum Flow Time	Unit Capacity Only?
[K73,ET75]	$\min\left(m^{3/2}, mn^{2/3}\right)$	Yes
[GR98]	$\tilde{O}\left(m^{3/2}\right)$	No
[M13]	$\tilde{O}(m^{10/7}U^{10/7})$	No
[LS14]	$\tilde{O}\left(m\sqrt{n}\right)$	No
[M16]	$\tilde{O}\left(m^{10/7}U^{1/7}\right)$	No

Use $\tilde{O}$ to hide $\log^{O(1)}(nU/\epsilon)$.

There is a $O(mn)$ strongly polynomial time algorithm, i.e. no appearance of U. [O13]

- $10/7 = 3/2 - 1/14$
- $11/8 = 3/2 - 1/8$
- $4/3 = 3/2 - 1/6$

Our Result

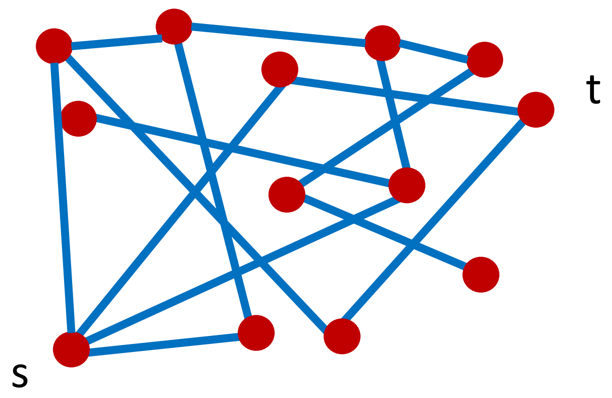
$m^{11/8+o(1)}U^{1/4}$

- Bipartite matching is $U = 1$ case
- Same runtime for minimum s-t cut

Undirected Flow Problems

Natural family of problems in combinatorial optimization.

- Graph $G = (V, E)$
- Vertices $s, t \in V$

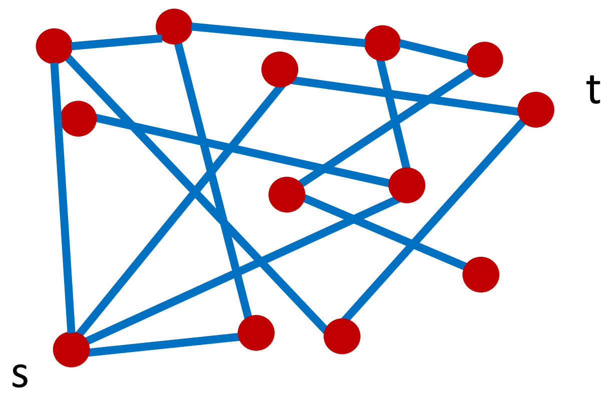


Undirected Flow Problems

Natural family of problems in combinatorial optimization.

What should we minimize?

- Graph $G = (V, E)$
- Vertices $s, t \in V$



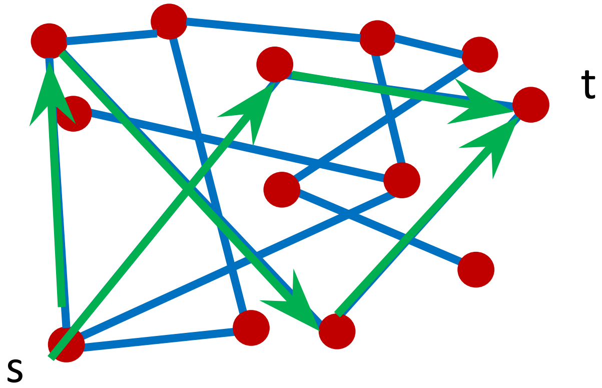
Goal
Send 1 unit of flow, $f \in \mathbb{R}^E$, between s and t in the "best" way possible.

Undirected Flow Problems

Natural family of problems in combinatorial optimization.

What should we minimize?

- Graph $G = (V, E)$
- Vertices $s, t \in V$



Goal
Send 1 unit of flow, $f \in \mathbb{R}^E$, between s and t in the "best" way possible.

Maximum Flow

$\tilde{O}(|E|\sqrt{|V|})$, $\tilde{O}(|E|^{10/7})$
[LS14] [M13]

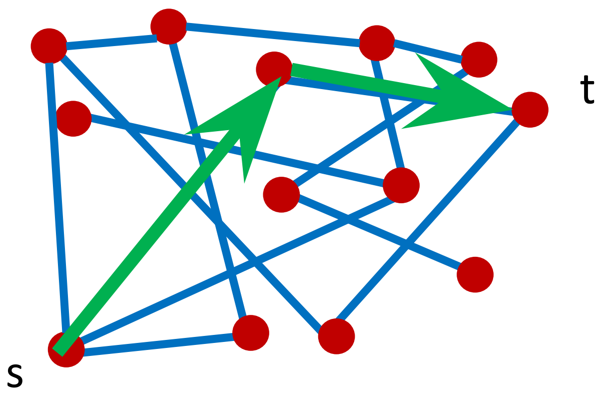
Congestion
$\max_{e \in E} |f_e|$

$\|f\|_\infty$

Undirected Flow Problems

Natural family of problems in combinatorial optimization.

- Graph $G = (V, E)$
- Vertices $s, t \in V$



Goal
Send 1 unit of flow, $f \in \mathbb{R}^E$, between s and t in the "best" way possible.

What should we minimize?

Shortest Path

$\tilde{O}(|E|)$

Length

$\sum_{e \in E} |f_e|$

$\|f\|_1$

Maximum Flow

$\tilde{O}(|E|\sqrt{|V|})$, $\tilde{O}(|E|^{10/7})$
[LS14] [M13]

Congestion

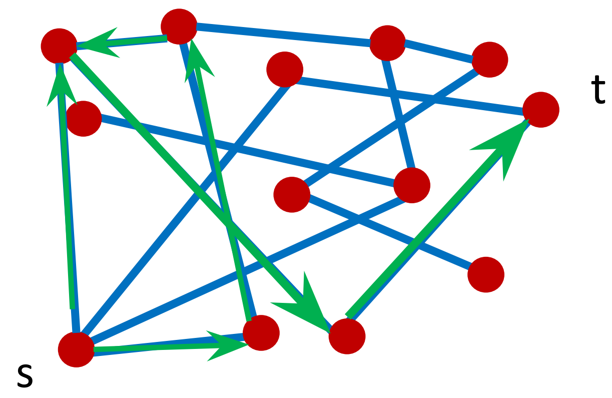
$\max_{e \in E} |f_e|$

$\|f\|_\infty$

Undirected Flow Problems

Natural family of problems in combinatorial optimization.

- Graph $G = (V, E)$
- Vertices $s, t \in V$



Goal
Send 1 unit of flow, $f \in \mathbb{R}^E$, between s and t in the "best" way possible.

What should we minimize?

Shortest Path

$\tilde{O}(|E|)$

Length
$\sum_{e \in E} |f_e|$

$\|f\|_1$

Electric Flow
Laplacian System Solving

$\tilde{O}(|E|)$
[ST04]

Energy
$\sum_{e \in E} |f_e|^2$

$\|f\|_2$

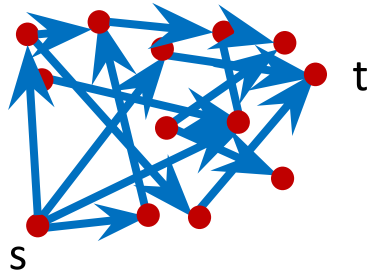
Maximum Flow

$\tilde{O}(|E|\sqrt{|V|})$, $\tilde{O}(|E|^{10/7})$
[LS14] [M13]

Congestion
$\max_{e \in E} |f_e|$

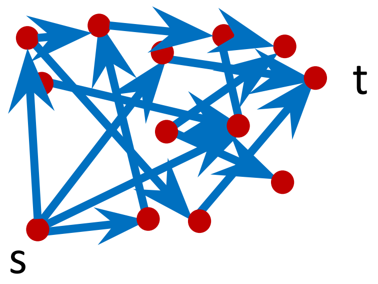
$\|f\|_\infty$

Running Times



- *Graph* $G = (V, E)$, $|V| = n$, $|E| = m$
- *Capacities* $u \in \{1, \ldots, U\}^E$
- *Source* $s \in V$ *and terminal* $t \in V$

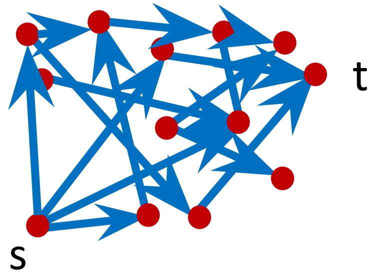
Running Times



- *Graph* $G = (V, E)$, $|V| = n$, $|E| = m$
- *Capacities* $u \in \{1, \dots, U\}^E$
- *Source* $s \in V$ *and terminal* $t \in V$

Authors	Exact Maximum Flow Time	Unit Capacity Only?
[K73,ET75]	$\min\left(m^{3/2}, mn^{2/3}\right)$	Yes
[GR98]	$\tilde{O}\left(m^{3/2}\right)$	No
[M13]	$\widetilde{O}(m^{10/7}U^{10/7})$	No
[LS14]	$\tilde{O}\left(m\sqrt{n}\right)$	No
[M16]	$\tilde{O}\left(m^{10/7}U^{1/7}\right)$	No

Running Times



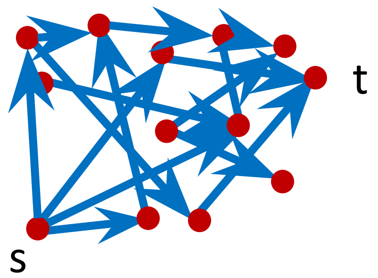
- *Graph* $G = (V, E)$, $|V| = n$, $|E| = m$
- *Capacities* $u \in \{1, \ldots, U\}^E$
- *Source* $s \in V$ *and terminal* $t \in V$

Authors	Exact Maximum Flow Time	Unit Capacity Only?
[K73,ET75]	$\min\left(m^{3/2}, mn^{2/3}\right)$	Yes
[GR98]	$\tilde{O}\left(m^{3/2}\right)$	No
[M13]	$\widetilde{O}(m^{10/7}U^{10/7})$	No
[LS14]	$\tilde{O}\left(m\sqrt{n}\right)$	No
[M16]	$\tilde{O}\left(m^{10/7}U^{1/7}\right)$	No

Augmenting Flows ([K73,ET75], [GR98])

Iterate on paths (ℓ_1-ish) problem

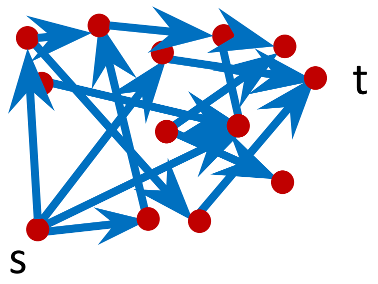
Running Times



- *Graph* $G = (V, E)$, $|V| = n$, $|E| = m$
- *Capacities* $u \in \{1, \dots, U\}^E$
- *Source* $s \in V$ *and terminal* $t \in V$

	Authors	Exact Maximum Flow Time	Unit Capacity Only?	
Augmenting Flows	[K73,ET75]	$\min\left(m^{3/2}, mn^{2/3}\right)$	Yes	*Iterate on paths (ℓ_1-ish) problem*
	[GR98]	$\tilde{O}\left(m^{3/2}\right)$	No	
Interior point methods [IPM]	[M13]	$\widetilde{O}(m^{10/7}U^{10/7})$	No	*Iterate on (ℓ_2) electric flows.*
	[LS14]	$\tilde{O}\left(m\sqrt{n}\right)$	No	
	[M16]	$\tilde{O}\left(m^{10/7}U^{1/7}\right)$	No	

Running Times



- *Graph* $G = (V, E)$, $|V| = n$, $|E| = m$
- *Capacities* $u \in \{1, \dots, U\}^E$
- *Source* $s \in V$ *and terminal* $t \in V$

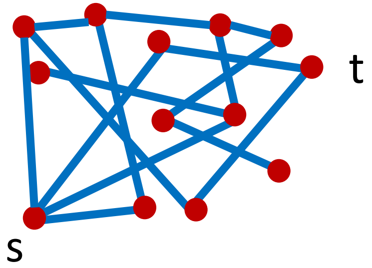
	Authors	Exact Maximum Flow Time	Unit Capacity Only?	
Augmenting Flows	[K73,ET75]	$\min\left(m^{3/2}, mn^{2/3}\right)$	Yes	*Iterate on paths (ℓ_1-ish) problem*
	[GR98]	$\tilde{O}\left(m^{3/2}\right)$	No	
Interior point methods [IPM]	[M13]	$\tilde{O}(m^{10/7}U^{10/7})$	No	*Iterate on (ℓ_2) electric flows.*
	[LS14]	$\tilde{O}\left(m\sqrt{n}\right)$	No	
	[M16]	$\tilde{O}\left(m^{10/7}U^{1/7}\right)$	No	

Iterate on something stronger?

Our Result

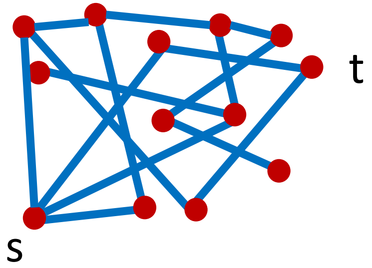
$m^{11/8+o(1)}U^{1/4}$

Undirected Graphs



ϵ-Approximate Flow
feasible $s \to t$ flow of value $(1-\epsilon)OPT$

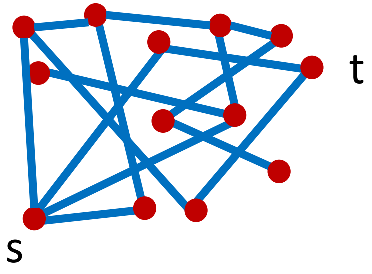
Undirected Graphs



ϵ-Approximate Flow
feasible $s \to t$ flow of value $(1-\epsilon)OPT$

[CKMST11]: $m^{4/3}\varepsilon^{-O(1)}$ runtime for (1-ε) approximate maxflow

Undirected Graphs

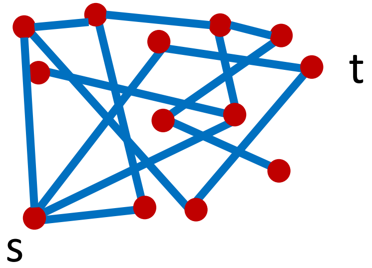


ϵ-Approximate Flow
feasible $s \to t$ flow of value $(1-\epsilon)OPT$

Uses electric flows (L2 minimizing flows)

[CKMST11]: $m^{4/3}\varepsilon^{-O(1)}$ runtime for (1-ε) approximate maxflow

Undirected Graphs



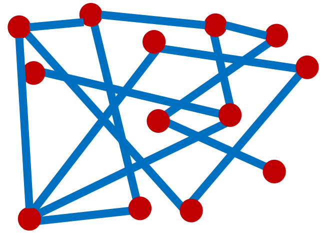
ϵ-Approximate Flow
feasible $s \to t$ flow of value $(1-\epsilon)OPT$

Uses electric flows (L2 minimizing flows)

[CKMST11]: $m^{4/3}\varepsilon^{-O(1)}$ runtime for (1-ε) approximate maxflow

[She13, KLOS14, Peng16, She18, ST18]: $m\varepsilon^{-1}$ runtime for (1-ε) approximate maxflow

Undirected Graphs



ϵ-Approximate Flow
feasible $s \to t$ flow of value $(1-\epsilon)OPT$

Uses electric flows (L2 minimizing flows)

[CKMST11]: $m^{4/3}\varepsilon^{-O(1)}$ runtime for $(1-\varepsilon)$ approximate maxflow

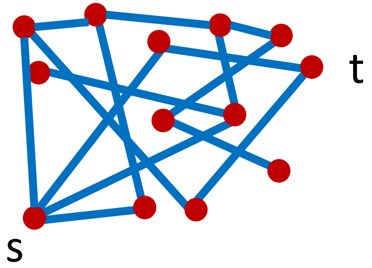
[She13, KLOS14, Peng16, She18, ST18]: $m\varepsilon^{-1}$ runtime for $(1-\varepsilon)$ approximate maxflow

How?
Work more directly in ℓ_∞.

Step 1
Build coarse ℓ_∞-approximator (e.g. oblivious routing or congestion approximator) to change representation.

Step 2
Apply iterative method to boost accuracy (e.g. gradient descent, mirror prox.)

Undirected Graphs



ϵ-Approximate Flow
feasible $s \to t$ flow of value $(1-\epsilon)OPT$

Uses electric flows (L2 minimizing flows)

[CKMST11]: $m^{4/3}\varepsilon^{-O(1)}$ runtime for (1-ε) approximate maxflow

[She13, KLOS14, Peng16, She18, ST18]: $m\varepsilon^{-1}$ runtime for (1-ε) approximate maxflow

Idea
Combine / apply these primitives in IPMs!

Problem
5-years and no luck ☹
IPM are ℓ_2, need dual.

How?
Work more directly in ℓ_∞.

Step 1
Build coarse ℓ_∞-approximator (e.g. oblivious routing or congestion approximator) to change representation.

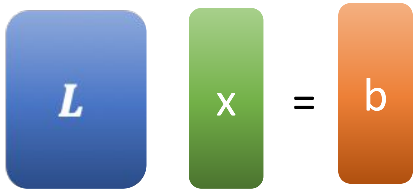
Step 2
Apply iterative method to boost accuracy (e.g. gradient descent, mirror prox.)

Stronger primitives?

Stronger primitives?

Directed Laplacians

Solve $\boldsymbol{L}x = b$ for
$\boldsymbol{L} = \boldsymbol{D}_{\text{out}}(G) - \boldsymbol{A}(G)$

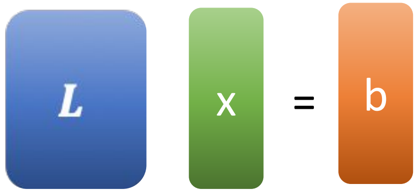


- Directed, asymmetric variant of electric flow and Laplacians systems.
- [CKPPSV16,CKPPRSV17,CKKPPRS18,AJSS19]
- Can solve in nearly linear time!
- PageRank, policy evaluation, stationary distribution computation, commute times, escape probabilities, Perron vectors in nearly linear time!

Stronger primitives?

Directed Laplacians

Solve $\boldsymbol{L}x = b$ for
$\boldsymbol{L} = \boldsymbol{D}_{\text{out}}(G) - \boldsymbol{A}(G)$



- Directed, asymmetric variant of electric flow and Laplacians systems.
- [CKPPSV16,CKPPRSV17,CKKPPRS18,AJSS19]
- Can solve in nearly linear time!
- PageRank, policy evaluation, stationary distribution computation, commute times, escape probabilities, Perron vectors in nearly linear time!

ℓ_p-Flows

$$\min_{\substack{unit\ s-t \\ flow\ f \in \mathbb{R}^E}} \|f\|_{r,p}^p = \sum_{e \in E} r_e |f_e|^p$$

- [BCLL18, AKPS19]
- Faster algorithms using electric flow ($p = 2$ case)
- Input sparsity runtimes!

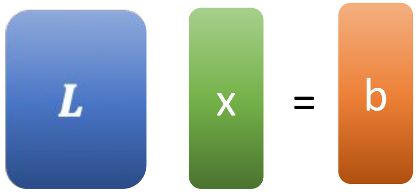
Stronger primitives?

Don't know how to use for directed maximum flow ☹

Don't know how to use to speed up IPMs ☹

Directed Laplacians

Solve $\boldsymbol{L}x = b$ for
$\boldsymbol{L} = \boldsymbol{D}_{\text{out}}(G) - \boldsymbol{A}(G)$



- Directed, asymmetric variant of electric flow and Laplacians systems.
- [CKPPSV16,CKPPRSV17,CKKPPRS18,AJSS19]
- Can solve in nearly linear time!
- PageRank, policy evaluation, stationary distribution computation, commute times, escape probabilities, Perron vectors in nearly linear time!

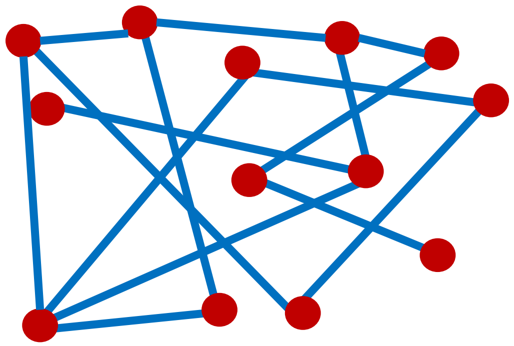
ℓ_p-Flows

$$\min_{\substack{unit\ s-t \\ flow\ f\in\mathbb{R}^E}} \|f\|_{r,p}^p = \sum_{e\in E} r_e |f_e|^p$$

- [BCLL18, AKPS19]
- Faster algorithms using electric flow ($p = 2$ case)
- Input sparsity runtimes!

Recent Advances in Flow Problems

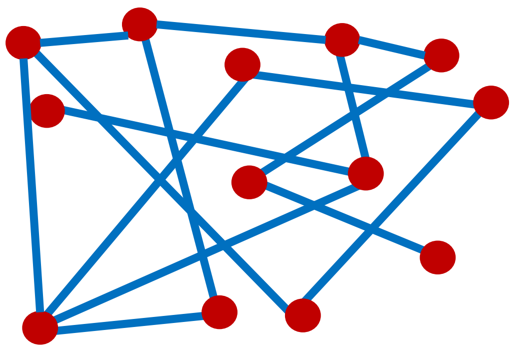
Part 1



Rest of Talk

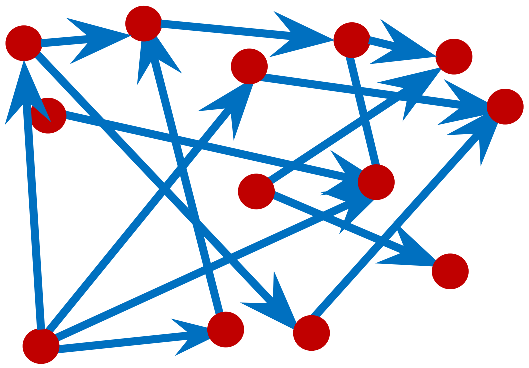
Recent Advances in Flow Problems

Part 1



[LS19] Energy Maximization and Maximum Flow

Part 2



<u>Directed Maximum Flow</u>

$m^{11/8+o(1)}U^{1/4}$

Strategy

Strategy

Step 0: Preprocessing

- Assume WLOG graph is undirected (i.e. $f_e \in [-u_e, u_e]$ for all $e \in E$)
- Precondition: add some edges from s to t
- Route until residual capacity is m^η and get exact solution in $\tilde{O}(m^{1+\eta})$

Strategy

Step 0: Preprocessing

- Assume WLOG graph is undirected (i.e. $f_e \in [-u_e, u_e]$ for all $e \in E$)
- Precondition: add some edges from s to t
- Route until residual capacity is m^η and get exact solution in $\tilde{O}(m^{1+\eta})$

Step 1: Interior Point Method

- Iteratively improve a s-t flow
- Improve objective (send more flow)
- Stay away from constraints (don't saturate edges)
- Pick potential to trade off each

Strategy

Step 0: Preprocessing

- Assume WLOG graph is undirected (i.e. $f_e \in [-u_e, u_e]$ for all $e \in E$)
- Precondition: add some edges from s to t
- Route until residual capacity is m^η and get exact solution in $\tilde{O}(m^{1+\eta})$

Step 1: Interior Point Method

- Iteratively improve a s-t flow
- Improve objective (send more flow)
- Stay away from constraints (don't saturate edges)
- Pick potential to trade off each

Step 2: Magic

Do better than naïve $\tilde{O}(\sqrt{m})$ iteration and $\tilde{O}(m^{3/2})$ runtime bound.

Madry 16 IPM Framework*

** Not exactly the framework but close.*

Madry 16 IPM Framework*

Algorithm State

- Feasible s-t flow $f \in \mathbb{R}^E$ of value v
- Forward weights $w^+ \in \mathbb{R}^E$
- Reverse weights $w^- \in \mathbb{R}^E$

** Not exactly the framework but close.*

Madry 16 IPM Framework*

Algorithm State

- Feasible s-t flow $f \in \mathbb{R}^E$ of value v
- Forward weights $w^+ \in \mathbb{R}^E$
- Reverse weights $w^- \in \mathbb{R}^E$

<u>Potential: Weighted Logarithmic Barrier</u>

$$\min_{\substack{s-t\ flow\ f\ of \\ value\ v}} V_w(f) = -\sum_{e \in E} \left(w_e^+ \log(u_e - f_e) + w_e^- \log(u_e + f_e) \right)$$

** Not exactly the framework but close.*

Madry 16 IPM Framework*

Algorithm State

- Feasible s-t flow $f \in \mathbb{R}^E$ of value v
- Forward weights $w^+ \in \mathbb{R}^E$
- Reverse weights $w^- \in \mathbb{R}^E$

Potential: Weighted Logarithmic Barrier

$$\min_{\substack{s-t\ flow\ f\ of \\ value\ v}} V_w(f) = -\sum_{e \in E} \left(w_e^+ \log(u_e - f_e) + w_e^- \log(u_e + f_e)\right)$$

Linear constraint: $B^\top f = v \cdot \chi_{s,t}$

** Not exactly the framework but close.*

Madry 16 IPM Framework*

Algorithm State

- Feasible s-t flow $f \in \mathbb{R}^E$ of value v
- Forward weights $w^+ \in \mathbb{R}^E$
- Reverse weights $w^- \in \mathbb{R}^E$

Potential: Weighted Logarithmic Barrier

$$\min_{\substack{s-t\ flow\ f\ of \\ value\ v}} V_w(f) = -\sum_{e\in E} \left(w_e^+ \log(u_e - f_e) + w_e^- \log(u_e + f_e)\right)$$

Linear constraint: $B^\top f = v \cdot \chi_{s,t}$

Penalizes saturating forward capacity.

** Not exactly the framework but close.*

Madry 16 IPM Framework*

Algorithm State

- Feasible s-t flow $f \in \mathbb{R}^E$ of value v
- Forward weights $w^+ \in \mathbb{R}^E$
- Reverse weights $w^- \in \mathbb{R}^E$

Potential: Weighted Logarithmic Barrier

$$\min_{\substack{s-t\ flow\ f\ of \\ value\ v}} V_w(f) = -\sum\nolimits_{e \in E} (w_e^+ \log(u_e - f_e) + w_e^- \log(u_e + f_e))$$

Linear constraint: $B^\top f = v \cdot \chi_{s,t}$

Penalizes saturating forward capacity.

Penalizes saturating reverse capacity.

** Not exactly the framework but close.*

Following Minimizers of the Log Barrier

$$\min_{B^\top f = v\cdot\chi_{s,t}} V_w(f) = -\sum\nolimits_{e\in E} \left(w_e^+ \log(u_e - f_e) + w_e^- \log(u_e + f_e)\right)$$

Following Minimizers of the Log Barrier

$$\min_{B^\top f = v \cdot \chi_{s,t}} V_w(f) = -\sum_{e \in E} \left(w_e^+ \log(u_e - f_e) + w_e^- \log(u_e + f_e) \right)$$

- **Centering step:** move from approximate minimizer to exact minimizer without changing value of v using electric flows.

Following Minimizers of the Log Barrier

$$\min_{B^\top f = v\cdot\chi_{s,t}} V_w(f) = -\sum_{e\in E}\left(w_e^+ \log(u_e - f_e) + w_e^- \log(u_e + f_e)\right)$$

- **Centering step:** move from approximate minimizer to exact minimizer without changing value of v using electric flows.
- **Progress step:** Add multiple of s-t electric flow with resistances given by Hessian of $V_w(f)$ to increase v to v+$\boldsymbol{\delta}$.

Following Minimizers of the Log Barrier

$$\min_{B^\top f = v\cdot\chi_{s,t}} V_w(f) = -\sum_{e\in E} \left(w_e^+ \log(u_e - f_e) + w_e^- \log(u_e + f_e)\right)$$

- **Centering step:** move from approximate minimizer to exact minimizer without changing value of v using electric flows.
- **Progress step:** Add multiple of s-t electric flow with resistances given by Hessian of $V_w(f)$ to increase v to v+$\boldsymbol{\delta}$.
- **Magic:** Change weights so progress steps can be larger!

Congestion Prevents Progress

Congestion Prevents Progress

- **When is the progress step forced to be small?**

Congestion Prevents Progress

- **When is the progress step forced to be small?**
- Intuitively when the s-t electric flow has large flow on some edges e compared to their residual capacities.

Congestion Prevents Progress

- **When is the progress step forced to be small?**
- Intuitively when the s-t electric flow has large flow on some edges e compared to their residual capacities.
- [CKMST11,M13,M16]: When electric flow is *congested* on an edge e, increasing resistance of edge e reduces electric flow on it.

Congestion Prevents Progress

- **When is the progress step forced to be small?**
- Intuitively when the s-t electric flow has large flow on some edges e compared to their residual capacities.
- [CKMST11,M13,M16]: When electric flow is *congested* on an edge e, increasing resistance of edge e reduces electric flow on it.
- Increase resistance by increasing weights w_e^+ and w_e^-.

Congestion Prevents Progress

- **When is the progress step forced to be small?**
- Intuitively when the s-t electric flow has large flow on some edges e compared to their residual capacities.
- [CKMST11,M13,M16]: When electric flow is *congested* on an edge e, increasing resistance of edge e reduces electric flow on it.
- Increase resistance by increasing weights w_e^+ and w_e^-.
- **Analysis:** Track electric flow energy as a potential function.

Congestion Prevents Progress

- **When is the progress step forced to be small?**
- Intuitively when the s-t electric flow has large flow on some edges e compared to their residual capacities.
- [CKMST11,M13,M16]: When electric flow is *congested* on an edge e, increasing resistance of edge e reduces electric flow on it.
- Increase resistance by increasing weights w_e^+ and w_e^-.
- **Analysis:** Track electric flow energy as a potential function.

How to improve?

New Approach: Energy Maximization

New Approach: Energy Maximization

- New *energy maximization* framework for computing weight changes

New Approach: Energy Maximization

- New *energy maximization* framework for computing weight changes
- **[M16]** combinatorial approach -- doubles resistances of edges that have large energy / electric flow.

New Approach: Energy Maximization

- New *energy maximization* framework for computing weight changes
- **[M16]** combinatorial approach -- doubles resistances of edges that have large energy / electric flow.
- **Our approach:** solve weight budgeted energy maximization as its own optimization problem!

Weight Increases via Energy Maximization

Weight Increases via Energy Maximization

- Let c_e be weight increase needed to increase r_e by 1
- Let W be weight change budget
- Maximum energy increase possible with weight change budget W:

Weight Increases via Energy Maximization

- Let c_e be weight increase needed to increase r_e by 1
- Let W be weight change budget
- Maximum energy increase possible with weight change budget W:

$$\max_{\|Cr'\|_1 \leq W} \text{energy}_{r+r'}(f) = \max_{\|Cr'\|_1 \leq W} \min_{B^\top f = \chi_{s,t}} \|f\|_{r+r',2}^2 = \min_{B^\top f = \chi_{s,t}} \max_{\|Cr'\|_1 \leq 1} \|f\|_{r+r',2}^2$$

Weight Increases via Energy Maximization

- Let c_e be weight increase needed to increase r_e by 1
- Let W be weight change budget
- Maximum energy increase possible with weight change budget W:

$$\max_{\|Cr'\|_1 \leq W} \text{energy}_{r+r'}(f) = \max_{\|Cr'\|_1 \leq W} \min_{B^\top f = \chi_{s,t}} \|f\|_{r+r',2}^2 = \min_{B^\top f = \chi_{s,t}} \max_{\|Cr'\|_1 \leq 1} \|f\|_{r+r',2}^2$$

$$= \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + \max_{\|Cr'\|_1 \leq W} \|f\|_{r',2}^2$$

Weight Increases via Energy Maximization

- Let c_e be weight increase needed to increase r_e by 1
- Let W be weight change budget
- Maximum energy increase possible with weight change budget W:

$$\max_{\|Cr'\|_1 \le W} \mathrm{energy}_{r+r'}(f) = \max_{\|Cr'\|_1 \le W} \min_{B^\top f = \chi_{s,t}} \|f\|_{r+r',2}^2 = \min_{B^\top f = \chi_{s,t}} \max_{\|Cr'\|_1 \le 1} \|f\|_{r+r',2}^2$$

$$= \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + \max_{\|Cr'\|_1 \le W} \|f\|_{r',2}^2$$

$$= \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + W \left\| C^{-1/2} f \right\|_\infty^2$$

Weight Increases via Energy Maximization

- Let c_e be weight increase needed to increase r_e by 1
- Let W be weight change budget
- Maximum energy increase possible with weight change budget W:

$$\max_{\|Cr'\|_1 \leq W} \text{energy}_{r+r'}(f) = \max_{\|Cr'\|_1 \leq W} \min_{B^\top f = \chi_{s,t}} \|f\|_{r+r',2}^2 = \min_{B^\top f = \chi_{s,t}} \max_{\|Cr'\|_1 \leq 1} \|f\|_{r+r',2}^2$$

$$= \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + \max_{\|Cr'\|_1 \leq W} \|f\|_{r',2}^2$$

$$= \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + W \left\|C^{-1/2} f\right\|_\infty^2$$

An undirected flow problem!!!!

Solving Energy Maximization Problem

The Energy Maximization Problem

$$\max_{\|Cr'\|_1 \leq W} \mathrm{energy}_{r+r'}(f) = \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + W \left\|C^{-1/2} f\right\|_\infty^2$$

Solving Energy Maximization Problem

The Energy Maximization Problem

$$\max_{\|Cr'\|_1 \leq W} \text{energy}_{r+r'}(f) = \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + W\left\|C^{-1/2} f\right\|_\infty^2$$

Exact Solve

- Pick $r = 0$
- Pick $c = 1/u_e^2$
- Is undirected maximum flow...

Solving Energy Maximization Problem

The Energy Maximization Problem

$$\max_{\|Cr'\|_1 \le W} \text{energy}_{r+r'}(f) = \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + W\left\|C^{-1/2} f\right\|_\infty^2$$

Exact Solve

- Pick $r = 0$
- Pick $c = 1/u_e^2$
- Is undirected maximum flow…

"Approximate" Solve

- Perform bounded weight increase for bound on ℓ_∞-norm of congestion.
- Take augmenting electric flow step
- Change weights to "center" highly congested edges (i.e. improve ℓ_4)
- Perform standard centering.

Smoothed ℓ_p Flows

Smoothed ℓ_p Flows

[AS19]
Also suffices for more ℓ_p-flow improvements.

Theorem [KPSW19] (informally)
For $p = \log^c n$ with $c \in (0,2/3)$ in $m^{1+o(1)}$ time can solve

$$\min_{\substack{unit\ s-t \\ flow\ f \in \mathbb{R}^E}} \|f\|_{r,2}^2 + \|f\|_p^p = \sum_{e \in E} r_e f_e^2 + \sum_{e \in E} |f_e|^p$$

Smoothed ℓ_p Flows

[AS19]
Also suffices for more ℓ_p-flow improvements.

Theorem [KPSW19] (informally)
For $p = \log^c n$ with $c \in (0,2/3)$ in $m^{1+o(1)}$ time can solve

$$\min_{\substack{unit\ s-t \\ flow\ f \in \mathbb{R}^E}} \|f\|_{r,2}^2 + \|f\|_p^p = \sum_{e \in E} r_e f_e^2 + \sum_{e \in E} |f_e|^p$$

Undirected Flow problem
No edge direction constraint.

Smoothed ℓ_p Flows

[AS19]
Also suffices for more ℓ_p-flow improvements.

Theorem [KPSW19] (informally)

For $p = \log^c n$ with $c \in (0,2/3)$ in $m^{1+o(1)}$ time can solve

$$\min_{\substack{unit\ s-t \\ flow\ f \in \mathbb{R}^E}} \|f\|_{r,2}^2 + \|f\|_p^p = \sum_{e \in E} r_e f_e^2 + \sum_{e \in E} |f_e|^p$$

Undirected Flow problem
No edge direction constraint.

Electric Potential
Weighted energy, ℓ_2

Smoothed ℓ_p Flows

[AS19]
Also suffices for more ℓ_p-flow improvements.

Theorem [KPSW19] (informally)

For $p = \log^c n$ with $c \in (0, 2/3)$ in $m^{1+o(1)}$ time can solve

$$\min_{\substack{unit\ s-t\\ flow\ f \in \mathbb{R}^E}} \|f\|_{r,2}^2 + \|f\|_p^p = \sum_{e \in E} r_e f_e^2 + \sum_{e \in E} |f_e|^p$$

Undirected Flow problem
No edge direction constraint.

Electric Potential
Weighted energy, ℓ_2

Maxflow-like Potential
Unweighted, high-power

How to maximizing energy efficiently?

The Energy Maximization Problem

$$\max_{\|Cr'\|_1 \le W} \mathrm{energy}_{r+r'}(f)$$
$$= \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + W\left\|C^{-1/2} f\right\|_\infty^2$$

How to maximizing energy efficiently?

The Energy Maximization Problem

$$\max_{\|Cr'\|_1 \le W} \mathrm{energy}_{r+r'}(f)$$

$$= \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + W \left\|C^{-1/2} f\right\|_\infty^2$$

Theorem [KPSW] (informally)

For $p = \log^c n$ with $c \in (0,2/3)$ in can solve in almost linear time:

$$\min_{\substack{unit\ s-t \\ flow\ f \in R^E}} \|f\|_{r,2}^2 + \|f\|_p^p$$

How to maximizing energy efficiently?

The Energy Maximization Problem

$$\max_{\|Cr'\|_1 \leq W} \text{energy}_{r+r'}(f)$$
$$= \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + W \left\|C^{-1/2} f\right\|_\infty^2$$

Theorem [KPSW] (informally)

For $p = \log^c n$ with $c \in (0, 2/3)$ in can solve in almost linear time:

$$\min_{\substack{unit\ s-t \\ flow\ f \in R^E}} \|f\|_{r,2}^2 + \|f\|_p^p$$

Problem

- $\|\cdot\|_\infty$ versus $\|\cdot\|_p$ for $p = \log^c n$
- Resistances versus flow
- $\|\cdot\|^2$ versus $\|\cdot\|^p$
- $C^{-1/2} f$ versus f

How to maximizing energy efficiently?

The Energy Maximization Problem

$$\max_{\|Cr'\|_1 \le W} \text{energy}_{r+r'}(f)$$

$$= \min_{B^\top f = \chi_{s,t}} \|f\|_{r,2}^2 + W \left\|C^{-1/2} f\right\|_\infty^2$$

Theorem [KPSW] (informally)

For $p = \log^c n$ with $c \in (0,2/3)$ in can solve in almost linear time:

$$\min_{\substack{unit\ s-t \\ flow\ f \in R^E}} \|f\|_{r,2}^2 + \|f\|_p^p$$

Problem

- $\|\cdot\|_\infty$ versus $\|\cdot\|_p$ for $p = \log^c n$
- Resistances versus flow
- $\|\cdot\|^2$ versus $\|\cdot\|^p$
- $C^{-1/2} f$ versus f

Solution

- $p = \log^c n$ is close enough for approx
- Standard primal-dual tricks
- Reducible by binary search
- … … …
- Pick $C = I$ and hope for the best?

Weight Reductions for Handling Unit lp Flows

Weight Reductions for Handling Unit lp Flows

Weighted Energy Max Algorithm

- Perform bounded weight increase for bound on ℓ_∞-norm of congestion.
- Take augmenting electric flow step
- Change weights to "center" highly congested edges (i.e. improve ℓ_4)
- Perform standard centering.

Weight Reductions for Handling Unit lp Flows

Weighted Energy Max Algorithm

- Perform bounded weight increase for bound on ℓ_∞-norm of congestion.
- Take augmenting electric flow step
- Change weights to "center" highly congested edges (i.e. improve ℓ_4)
- Perform standard centering.

Unweighted Energy Max Algorithm

Undo un-necessary weight changes

Tradeoff yields $m^{11/8+o(1)}U^{1/4}$

Weight Reductions for Handling Unit lp Flows

Weighted Energy Max Algorithm

- Perform bounded weight increase for bound on ℓ_∞-norm of congestion.
- Take augmenting electric flow step
- Change weights to "center" highly congested edges (i.e. improve ℓ_4)
- Perform standard centering.

Unweighted Energy Max Algorithm

Undo un-necessary weight changes

Tradeoff yields $m^{11/8+o(1)}U^{1/4}$

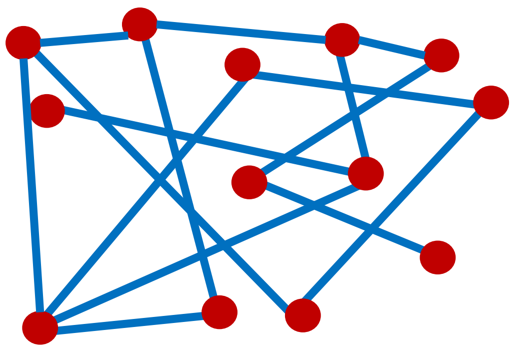
No potential function!

Note

Can get $m^{7/5+o(1)}U^{2/5}$ even without this.

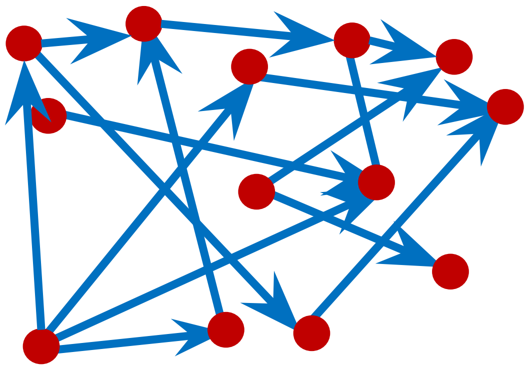
Recent Advances in
Flow Problems

Part 1



[LS19] Energy Maximizatio
and Maximum Flow

Part 2



Directed Maximum Flow

$m^{11/8+o(1)}U^{1/4}$

Future Directions / Open Problems

Future Directions / Open Problems

Better Primitives?

- Smoothed lp flows for weighted graphs

Future Directions / Open Problems

Better Primitives?

- Smoothed lp flows for weighted graphs

Better Methods?

- Can we go beyond IPMs?
- Can we completely leave l2 as with approximate maxflows?
- Can we more directly use smoothed l2-lp flows?

Future Directions / Open Problems

Better Primitives?

- Smoothed lp flows for weighted graphs

Better Methods?

- Can we go beyond IPMs?
- Can we completely leave l2 as with approximate maxflows?
- Can we more directly use smoothed l2-lp flows?

Better Combinatorics?

- Achieving a (log U) dependence.

Future Directions / Open Problems

Better Primitives?

- Smoothed lp flows for weighted graphs

Better Methods?

- Can we go beyond IPMs?
- Can we completely leave l2 as with approximate maxflows?
- Can we more directly use smoothed l2-lp flows?

Better Combinatorics?

- Achieving a (log U) dependence.

New Result! [LS20]
$m^{4/3+o(1)} U^{1/3}$

Future Directions / Open Problems

Better Primitives?

- Smoothed lp flows for weighted graphs

Better Methods?

Divergence minimizing flows

- Can we go beyond IPMs?
- Can we completely leave l2 as with approximate maxflows?
- Can we more directly use smoothed l2-lp flows?

Better Combinatorics?

- Achieving a (log U) dependence.

New Result! [LS20]

$m^{4/3+o(1)}U^{1/3}$

Future Directions / Open Problems

Better Primitives?

- Smoothed lp flows for weighted graphs

Better Methods?

Divergence minimizing flows

- Can we go beyond IPMs?
- Can we completely leave l2 as with approximate maxflows?
- Can we more directly use smoothed l2-lp flows?

Regularized Newton Steps

Better Combinatorics?

- Achieving a (log U) dependence.

New Result! [LS20]
$m^{4/3+o(1)}U^{1/3}$

Future Directions / Open Problems

Better Primitives?

- Smoothed lp flows for weighted graphs

Better Methods?

- Can we go beyond IPMs?
- Can we completely leave l2 as with approximate maxflows?
- Can we more directly use smoothed l2-lp flows?

Divergence minimizing flows

Regularized Newton Steps

Better Combinatorics?

- Achieving a (log U) dependence.

New Result! [LS20]
$m^{4/3+o(1)}U^{1/3}$

[AMV20] Mincost flows in time $m^{4/3+o(1)}\log C$

Faster Energy Maximization for Faster Maximum Flow

arXiv : 1910.14276

The End

Questions?

Yang P. Liu

Aaron Sidford

Contact Info:

- email: *yangpliu@stanford.edu*
- website: *yangpliu.github.io*