Parallel Repetition for k-Player Projection Games

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k-Player Game

- Finite questions sets $X_1, X_2, \dots, X_k$ and answer sets $A_1, A_2, \dots, A_k$
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- Strategies: functions $f_i: X_i \to A_i$ for $i = 1, 2, \dots, k$
- Winning probability: $\mathbb{E}_{(x_1,\dots,x_k)\sim\mu} V(x_1, \dots, x_k, f(x_1), \dots, f(x_k))$ for arbitrary known predicate $V: X_1 \times \cdots \times X_k \times A_1 \times \cdots \times A_k \to \{0, 1\}$

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- Game $\mathcal{G}$ consists of μ, V
- $val(\mathcal{G})$ denotes maximum winning probability

Example 3XOR Game

- System of linear equations $x_{i_t} + y_{j_t} + z_{k_t} = a_t$, say in $\mathbb{F}_2$
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- Winning probability: maximum fraction of equations that can be satisfied simultaneously

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- Call this game $\mathcal{G}^{\otimes n}$: does $val(\mathcal{G}^{\otimes n})$ decay (exponentially) with n?

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- Many follow ups:
  - Simpler proofs and improved bounds [Holenstein09,Rao11,DS14]
  - Better understanding for why $val(\mathcal{G}^{\otimes n}) = val(\mathcal{G})^n$ fails

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  - GHZ game [Braverman-Khot-Minzer]
- Connected games: consider graph where two inputs $x \in X_1 \times \cdots \times X_k$ and $x' \in X_1 \times \cdots \times X_k$ have an edge if x, x' differ in one coordinate
- [Dinur-Harsha-Venkat-Yuen]: $val(\mathcal{G}^{\otimes n})$ decays exponentially for connected games

k-Player Projection Games

- For each question $x \in supp(\mu)$, there are maps $\sigma_x^i: A_i \rightarrow D_x$ for some set D_x such that answer $(a_1, \ldots, a_k)$ wins if and only if
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$$\sigma_x^1(a_1) = \sigma_x^2(a_2) = \cdots = \sigma_x^k(a_k)$$
- Equivalently, for a fixed question x, the k-partite hypergraph on $A_1 \times \cdots \times A_k$ formed by accepting answers satisfies that: all connected components are complete hypergraphs

Main Result and Outline

- **Theorem:** If $\mathcal{G}$ is a k-player projection game with $\mathrm{val}(\mathcal{G}) < 1$, then $\mathrm{val}\left(\mathcal{G}^{\otimes n}\right) \leq \exp(-\Omega_{\mathcal{G}}(n))$

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- **Theorem:** If $\mathcal{G}$ is a k-player projection game with $\mathrm{val}(\mathcal{G}) < 1$, then $\mathrm{val}\left(\mathcal{G}^{\otimes n}\right) \leq \exp(-\Omega_{\mathcal{G}}(n))$
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- **Theorem:** If $\mathcal{G}$ is a k-player projection game with $\mathrm{val}(\mathcal{G}) < 1$, then $\mathrm{val}\left(\mathcal{G}^{\otimes n}\right) \leq \exp(-\Omega_{\mathcal{G}}(n))$
- **Proof plan:** Iteratively increase the support size of $\mathcal{G}$ until its connected.
- **Precisely:** Construct a game $\mathcal{G}'$ such that $\mathrm{supp}(\mathcal{G}) \subset \mathrm{supp}(\mathcal{G}')$,
$$\mathrm{val}\left(\mathcal{G}^{\otimes n}\right)^{O(1)} \leq \mathrm{val}\left({\mathcal{G}'}^{\otimes n}\right)$$
- Apply parallel repetition for connected games [DHVY'17]

Increasing the Support

- Consider 3-player games for simplicity
- Define a new game $\mathcal{G}'$ as follows:
  - Sample two questions (x, y, z) and (x, y', z') which share the value of x
  - Send (x, y, z') to the players and receive answers
  - If $(x, y, z) \neq (x, y', z')$ always accept
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  - If $(x, y, z) = (x, y', z')$ accept based on (x, y, z)
- Note: $\mathcal{G}'$ is a projection game
- Need: relate $\mathrm{val}\left(\mathcal{G}'^{\otimes n}\right)$ to $\mathrm{val}(\mathcal{G}^{\otimes n})$

Relating Game Values

- Probability that both (x, y, z) and (x, y', z') are winning in $\mathcal{G}$ is at least $\mathrm{val}\left(\mathcal{G}^{\otimes n}\right)^2$ by Cauchy-Schwarz
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- Thus $\mathrm{val}\left({\mathcal{G}'}^{\otimes n}\right) \geq \mathrm{val}\left(\mathcal{G}^{\otimes n}\right)^2$
- Apply this transformation repeatedly (swap roles of x, y, z)
- Eventually game is connected

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- Proof slowly increases support size until the game is connected

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- Proof slowly increases support size until the game is connected
- Other classes of games with strong parallel repetition results?
- “Barriers” to applying this method to general parallel repetition?

The End