Parallel Reachability in Square Root Depth

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Stanford University

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Stanford University

Reachability Problem in Directed Graphs

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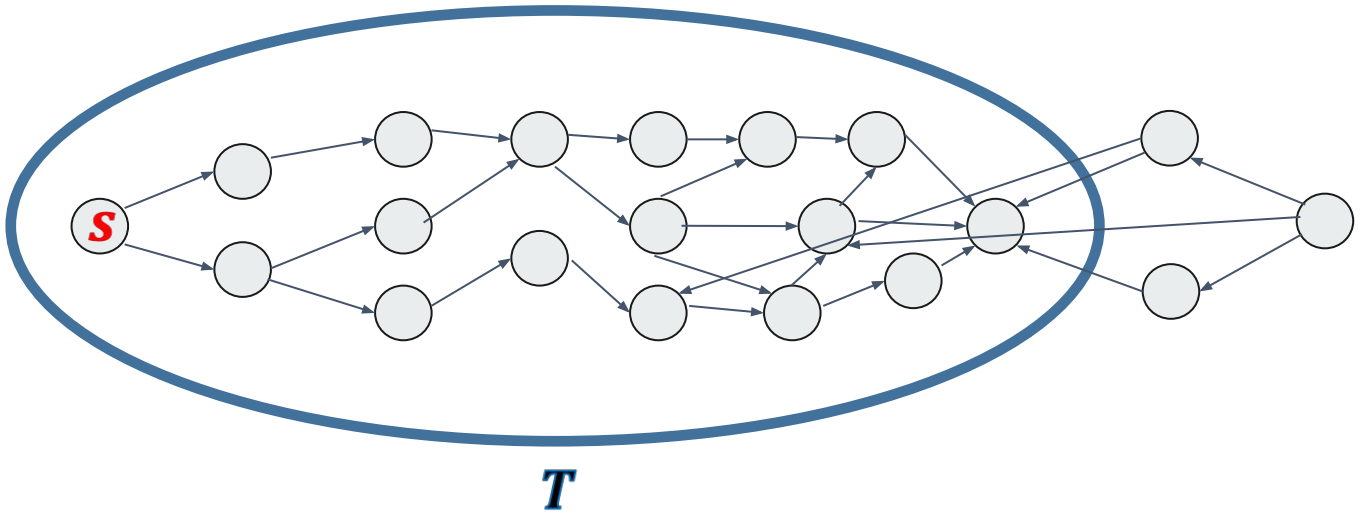

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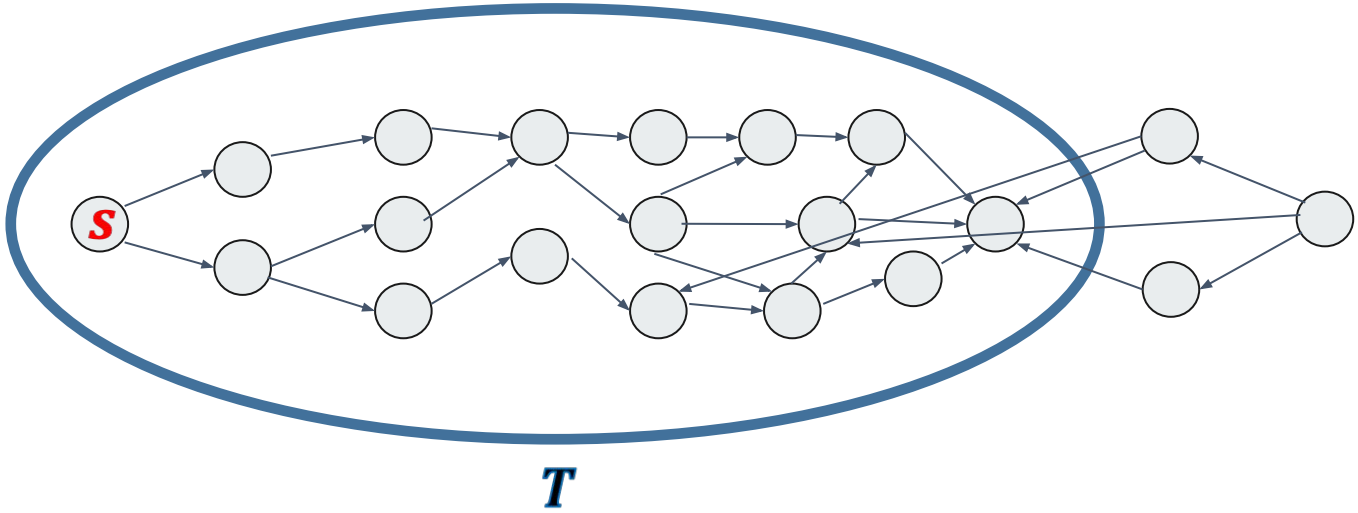

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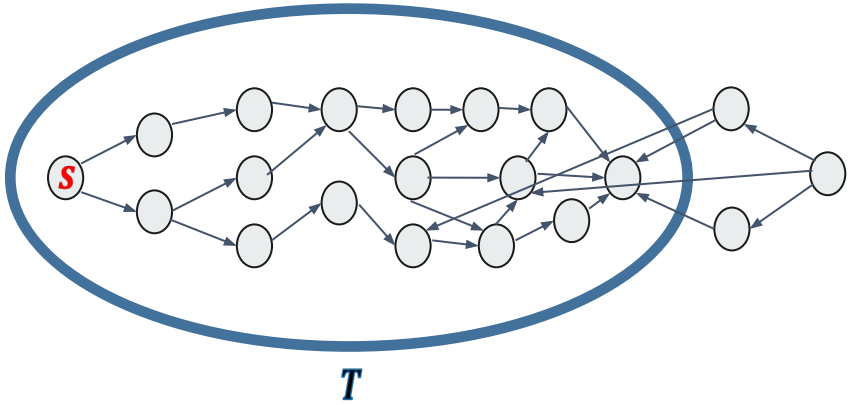
Why?

- Simplest directed graph problem
- Barrier towards efficient algorithms when edge asymmetry



Work on Parallel Reachability Before 2017

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Algorithm	**Work**	**Depth**
Parallel BFS	O(m)	O(D)
Parallel Trans. Closure	$O(n^{\omega})$	$\tilde{O}(1)$
Spencer's [Spe97]	$\tilde{O}(m + n\rho^2)$	$\tilde{O}(n/\rho)$
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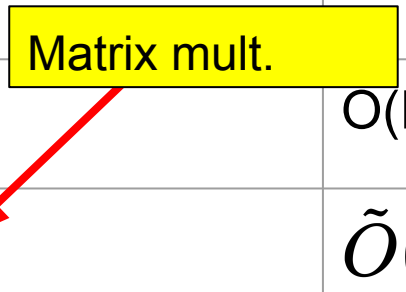

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None are linear work and sublinear depth.

Shortcutting and Hopsets

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Hopset: such a set of edges

Shortcut: edge in a hopset

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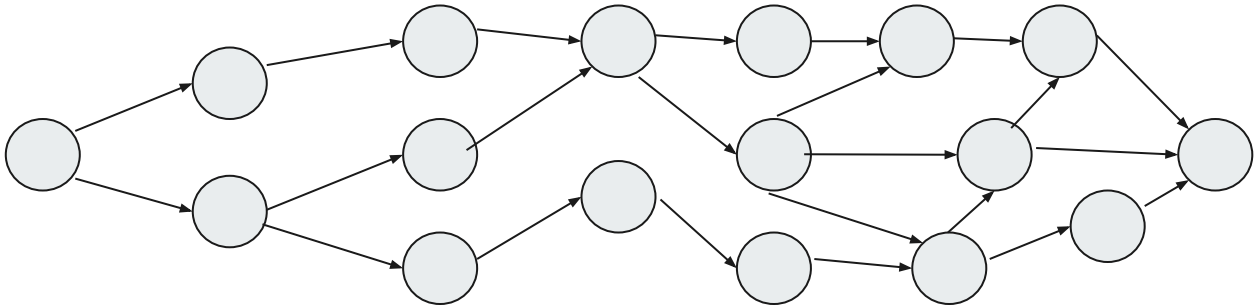

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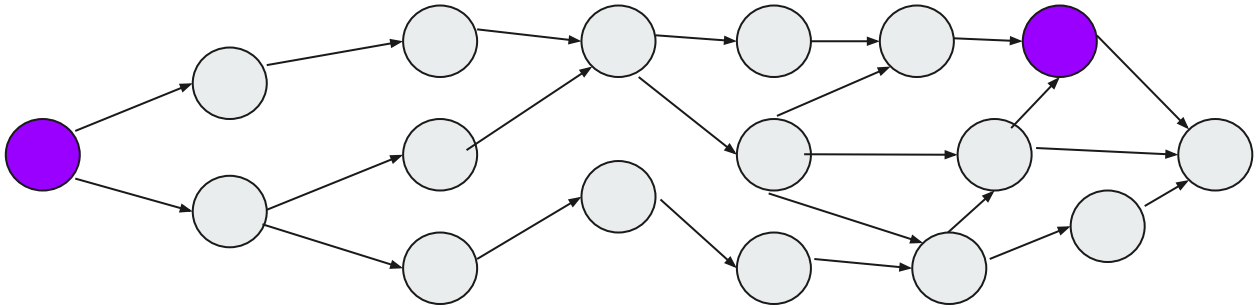

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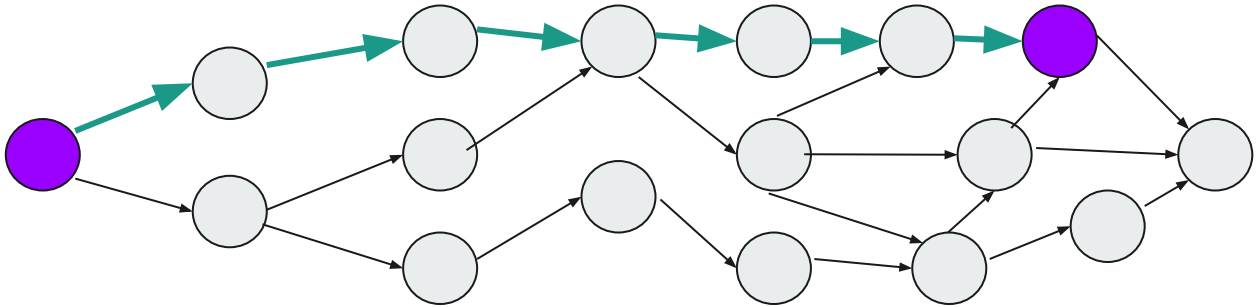

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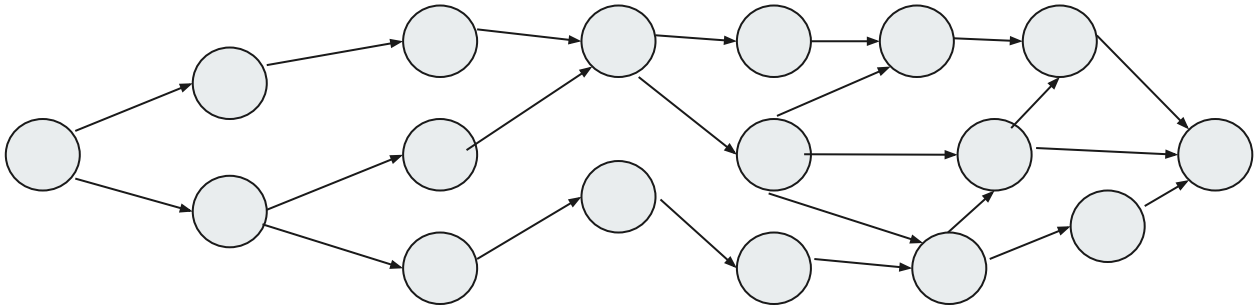

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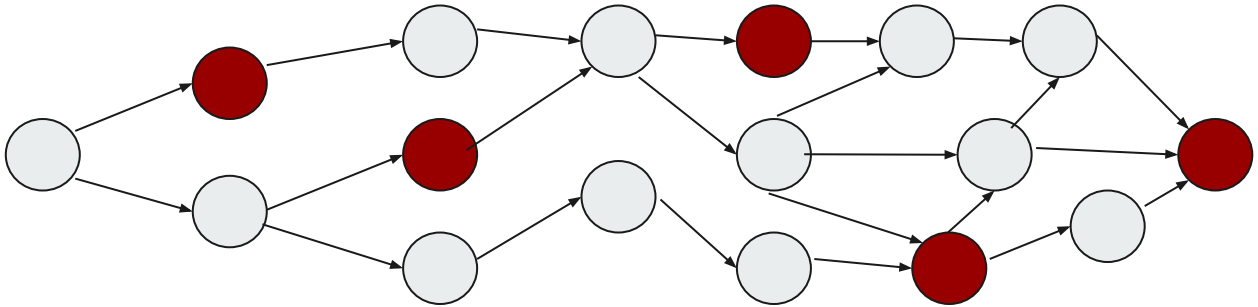

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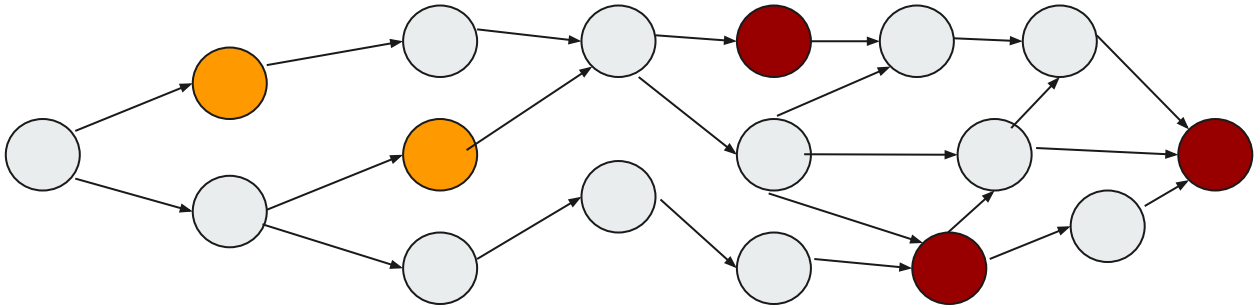

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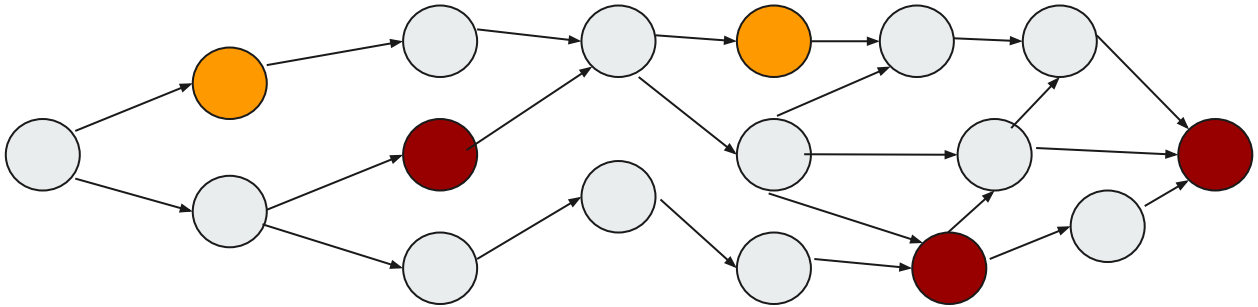

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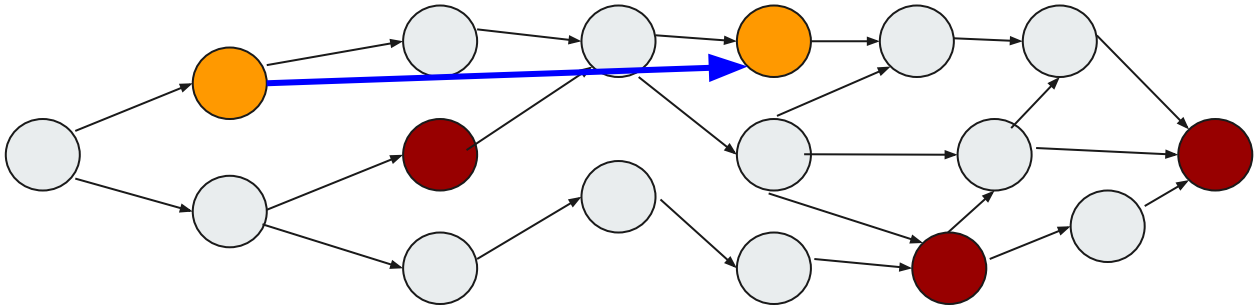

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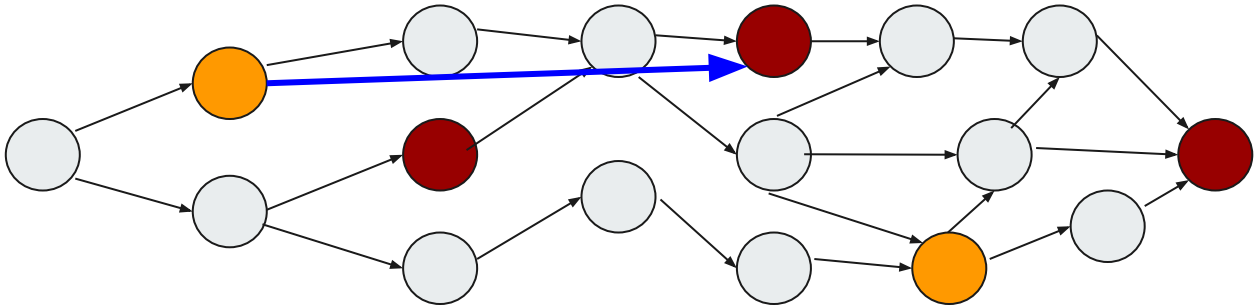

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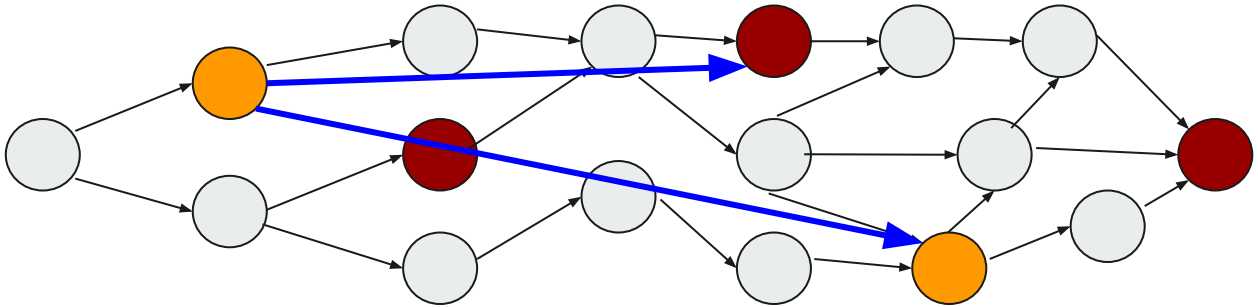

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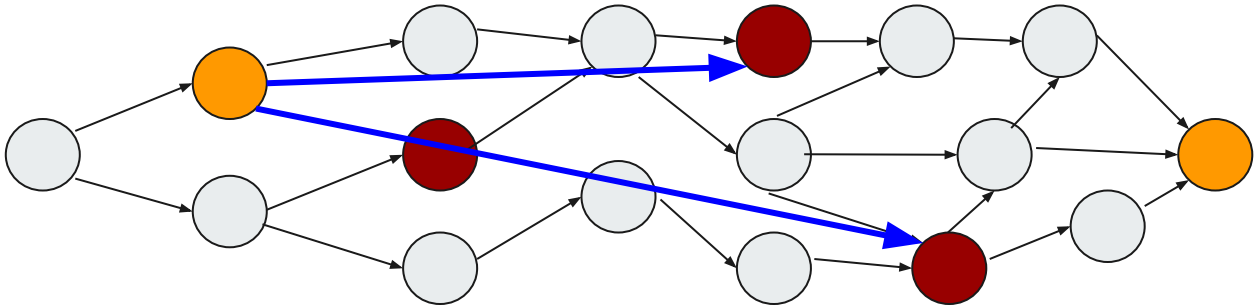

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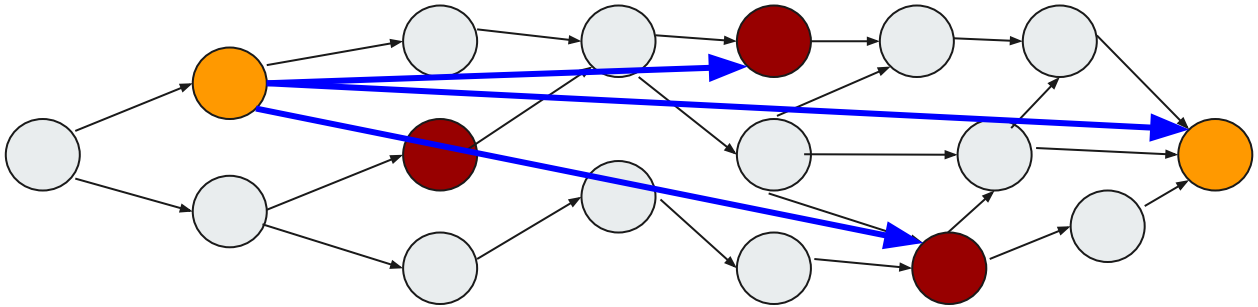

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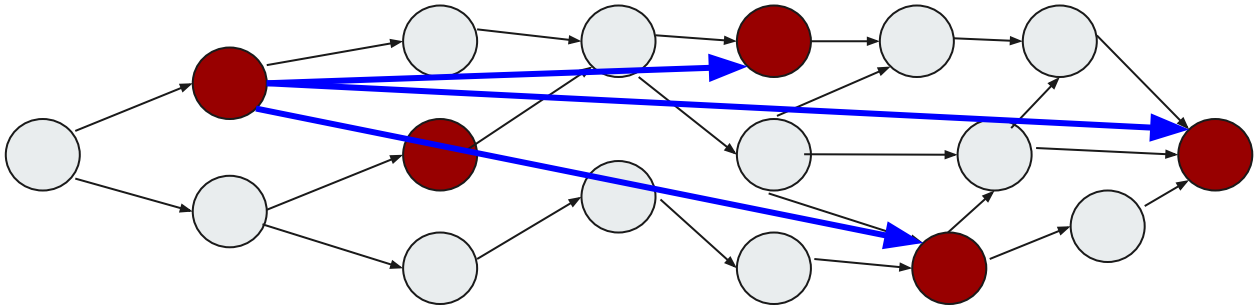

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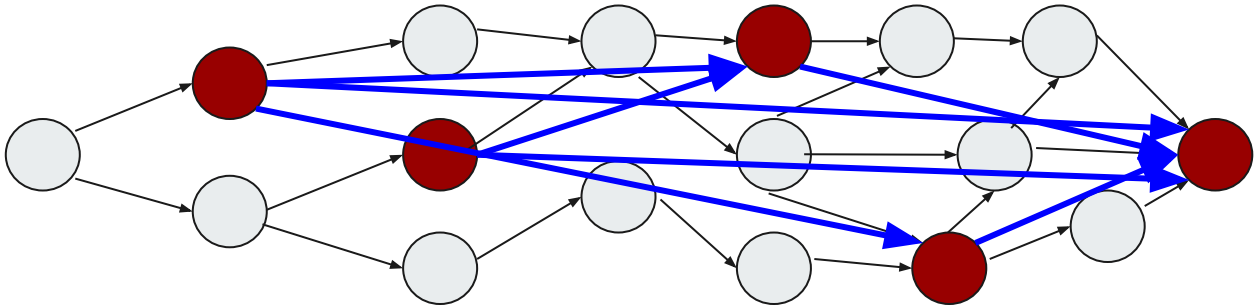

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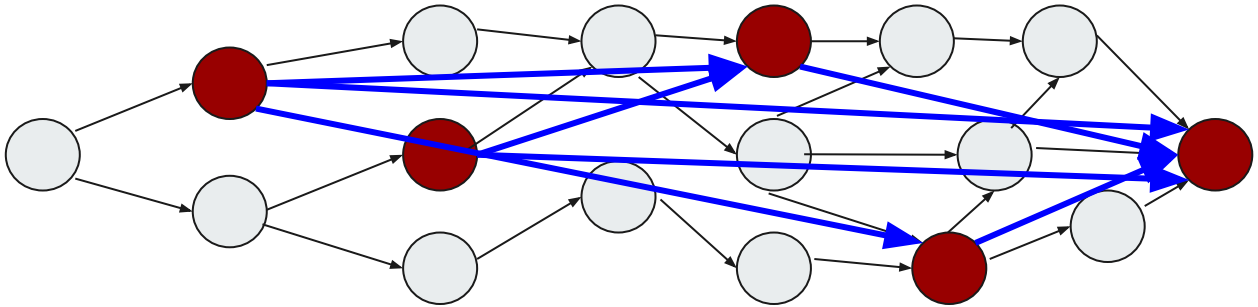

Blue edges form a hopset.

Each blue edge is a shortcut.

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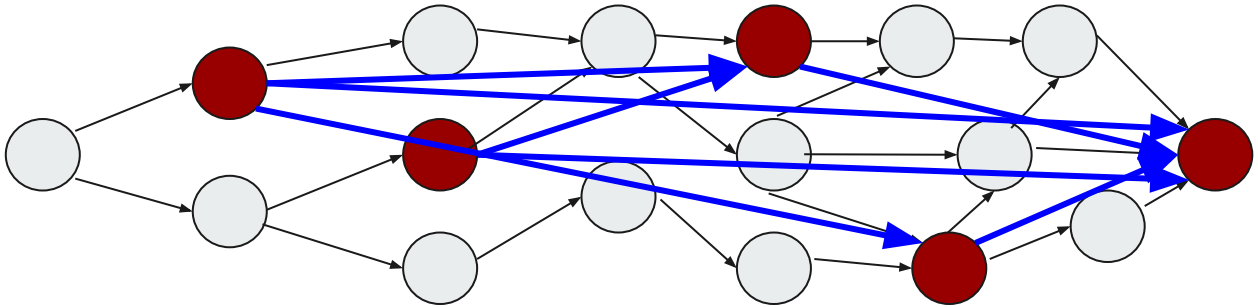

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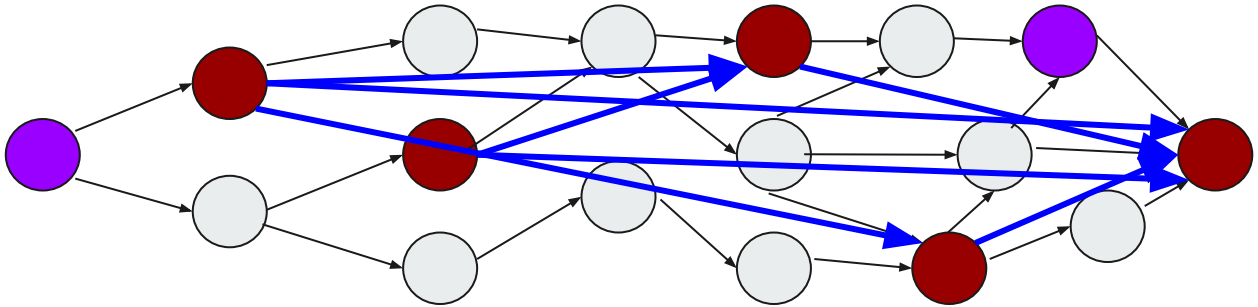

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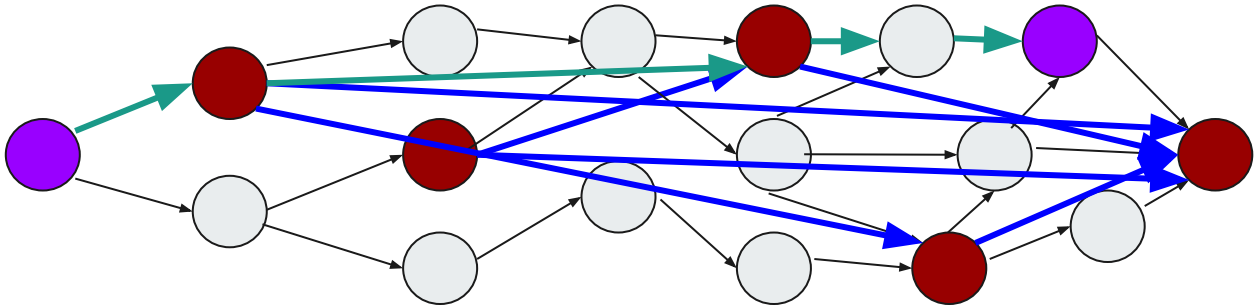

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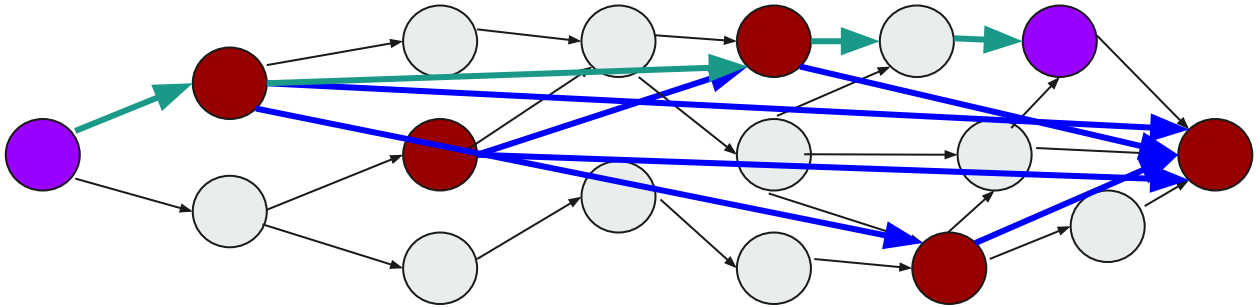

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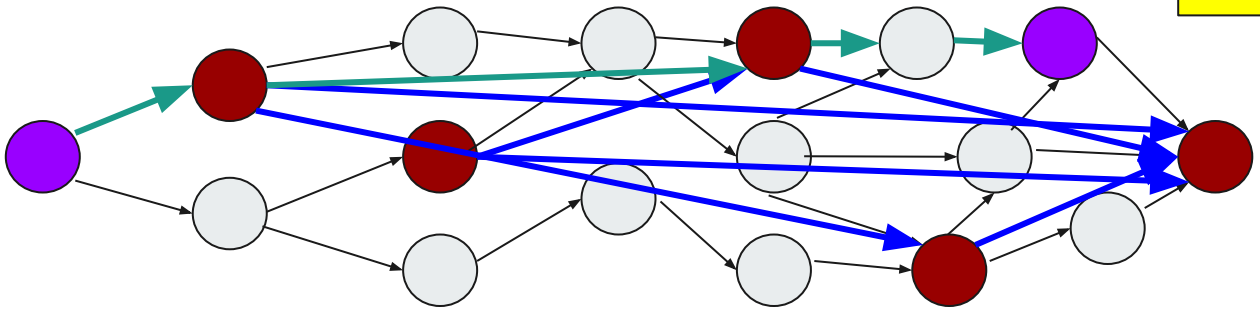
Good depth, but superlinear work.



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Best known *combinatorial construction* (even ignoring algorithmic efficiency)

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Theorem [Fin18]: There is an algorithm which computes a hopset with $\tilde{O}(n)$ edges which reduces the diameter to $\tilde{O}(n^{2/3})$.
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Can we match the *hopset bound*, i.e. nearly linear work and depth $O(\sqrt{n})$?

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Theorem [JLS19]: There is an algorithm which computes a hopset with $\tilde{O}(nk)$ edges which reduces the diameter to $n^{1/2+O(1/\log k)}$.
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Each vertex is a processor with infinite computational power.

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Directed graph G.

Each vertex is a processor with infinite computational power.

Vertices only know their in and out neighbors.

Each round, a vertex can send a message of length O(log n) to each of its in and out neighbors (not necessarily the same message).

Distributed Computation

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Complexity depends on n (number of vertices) and D (undirected diameter of the underlying graph G)

Distributed Reachability Problem

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s does not have to know which vertices it can reach

Every vertex knows s to start

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Theorem [DSHK+11]: Any algorithm for reachability in the **CONGEST** model requires $\tilde{\Omega}(D + \sqrt{n})$ rounds.

Distributed Reachability: What's Known

Distributed Reachability: Our Results

Theorem [JLS19]: There is an algorithm for reachability in the **CONGEST** model which takes $\tilde{O}(n^{1/2} + n^{1/3+o(1)} D^{2/3})$ rounds.

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Theorem [JLS19]: There is an algorithm for reachability in the **CONGEST** model which takes $\tilde{O}(n^{1/2} + n^{1/3+o(1)} D^{2/3})$ rounds.

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First algorithm matching the lower bound for $D = \Omega(n^{\delta})$ for some $\delta > 0$

Our algorithm in the **CONGEST** model is a fairly general way of converting "reasonable" low work and depth parallel reachability algorithms into distributed algorithms.

Using Fineman's algorithm instead of our algorithm still improves over the previous state of the art.

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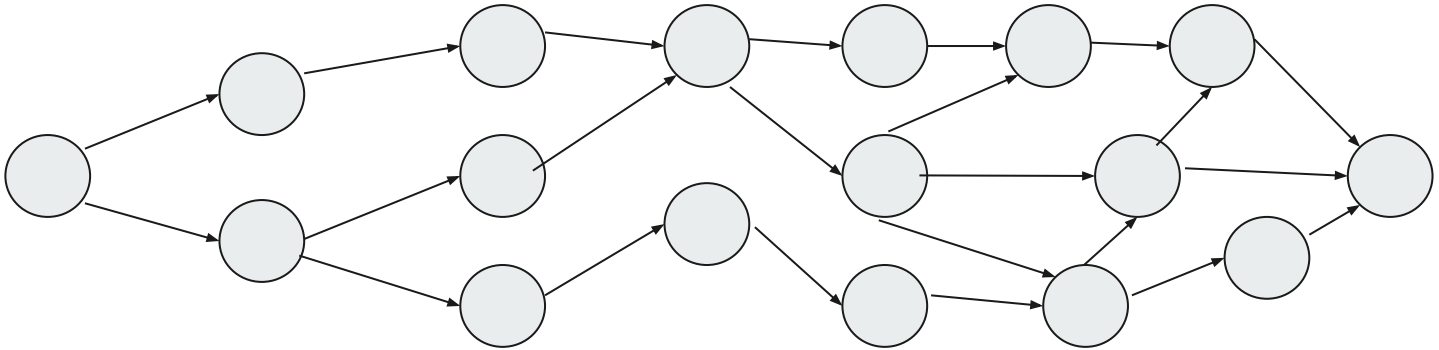
How to Construct Hopsets?

We will ignore strongly connected components.

We first give a serial linear-time construction. We'll discuss parallelization later.



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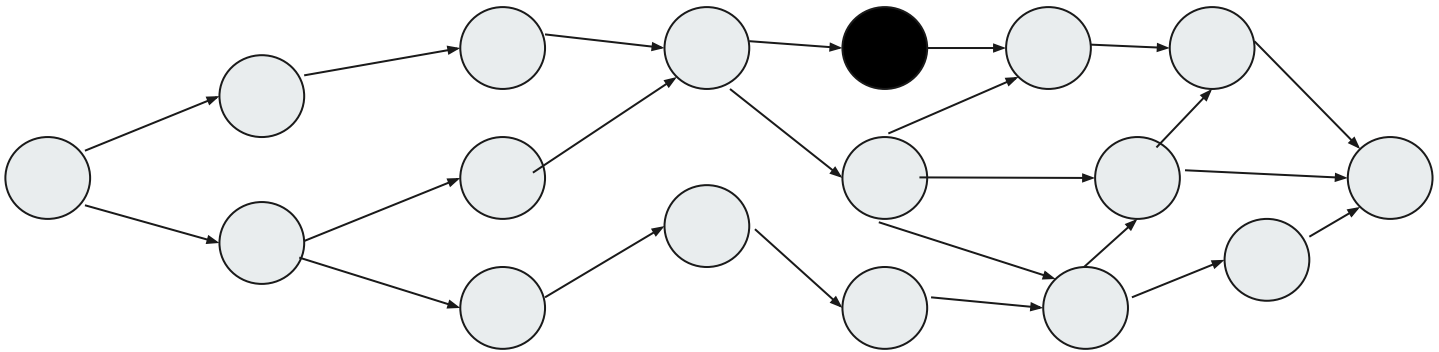

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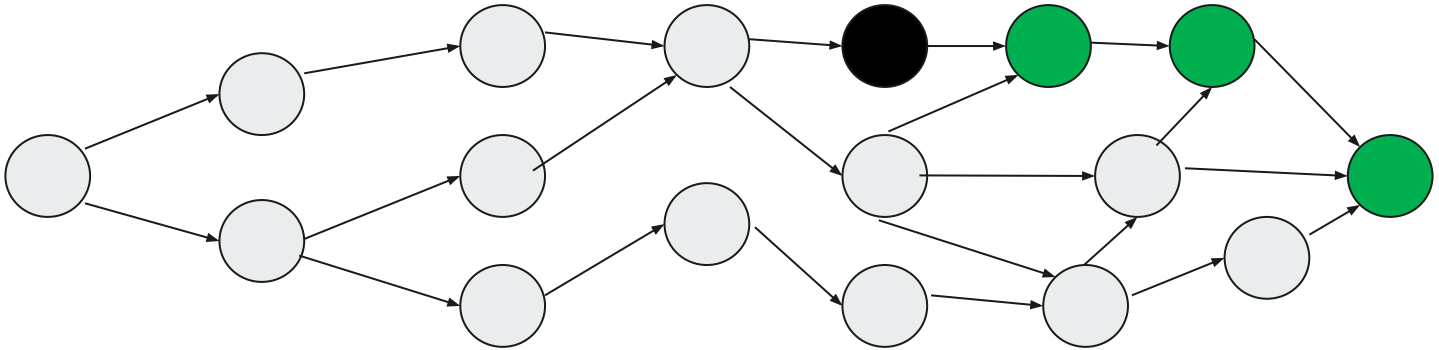
Add a shortcut from **v** to every vertex it can reach (**D = Descendant**)



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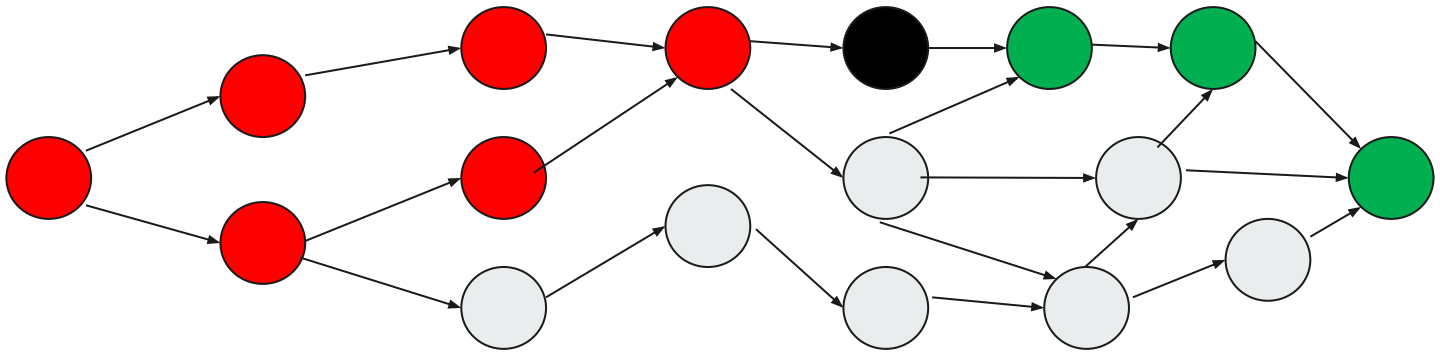

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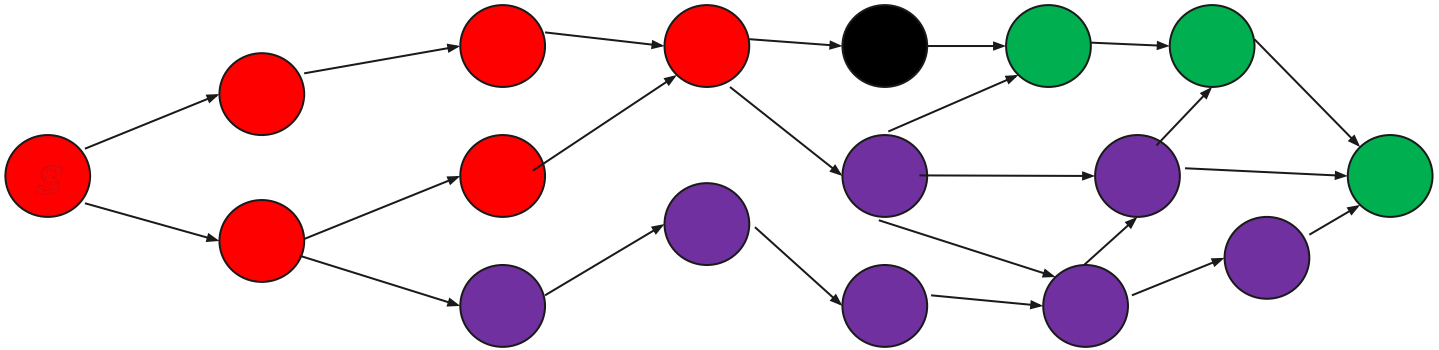
Recurse on **D**, **A**, and **U = V \ D \ A = Unrelated**.



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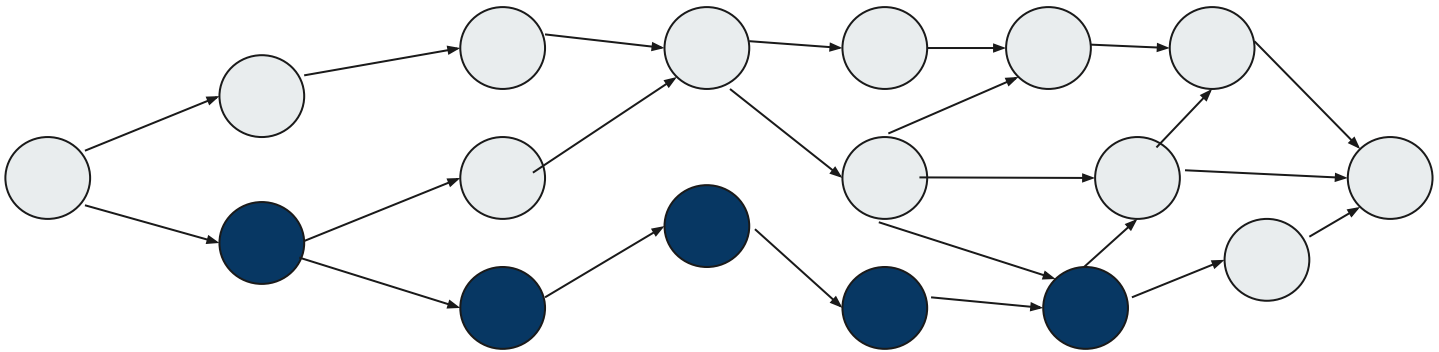

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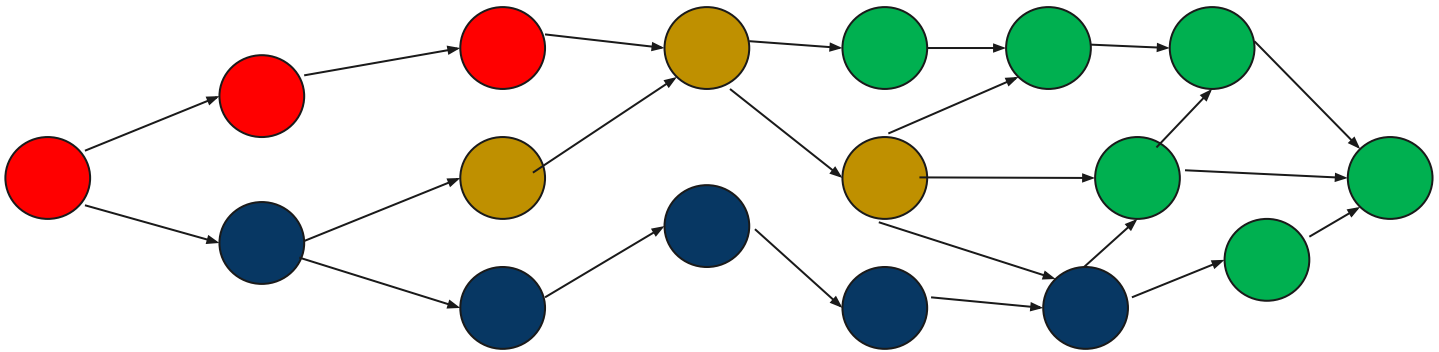

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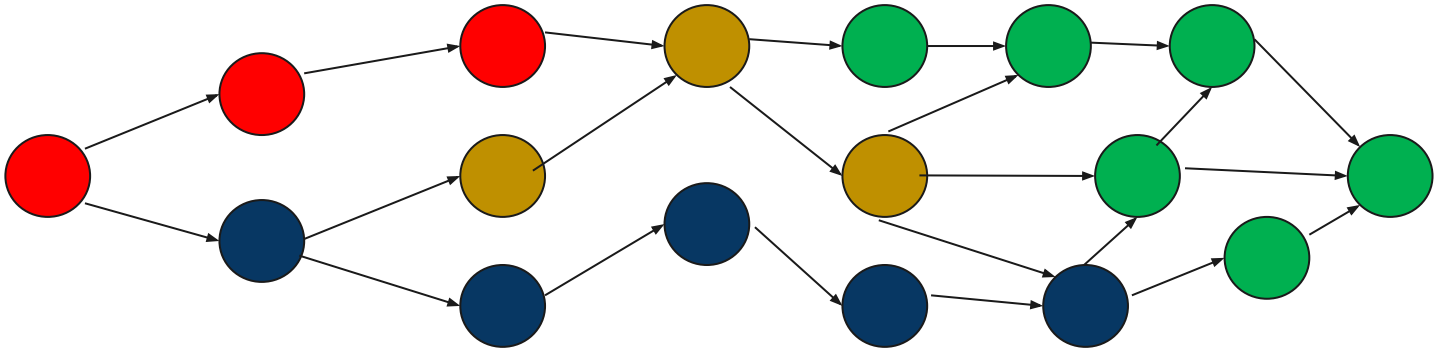

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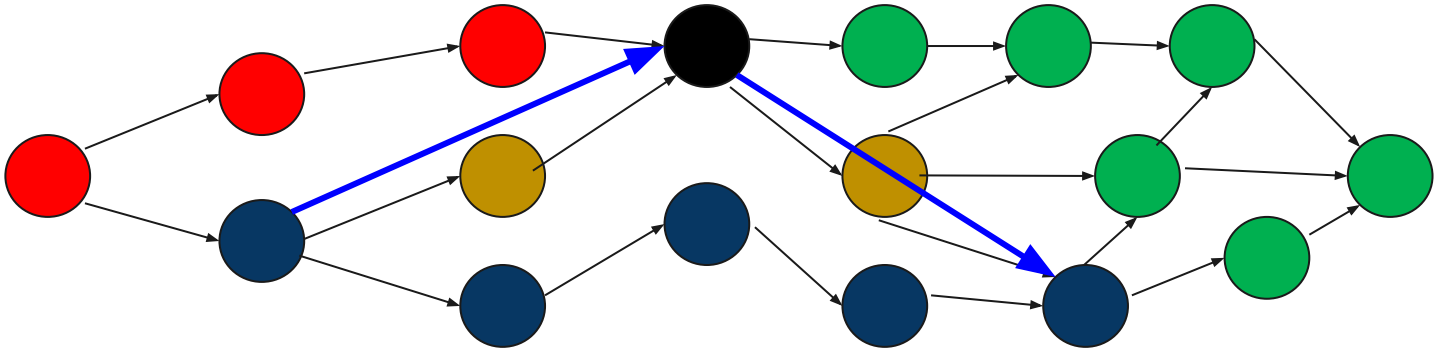

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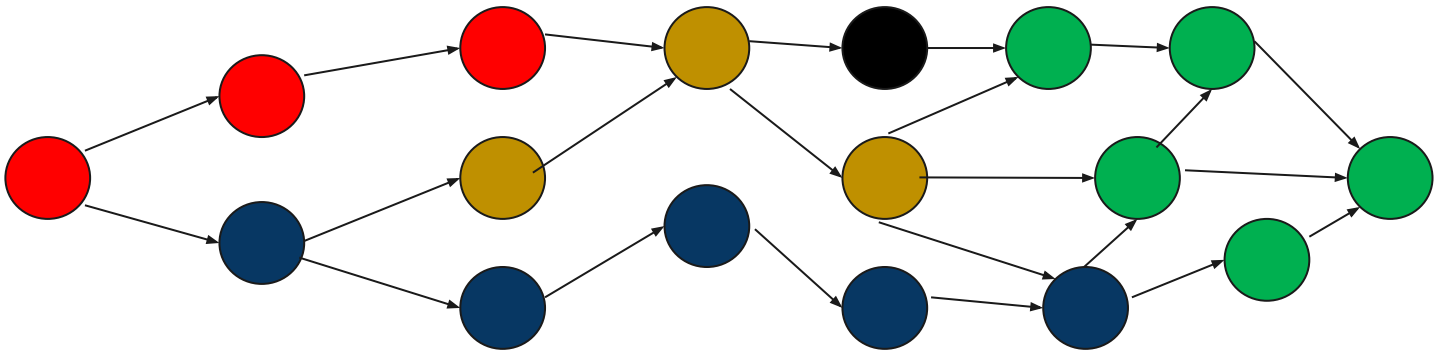

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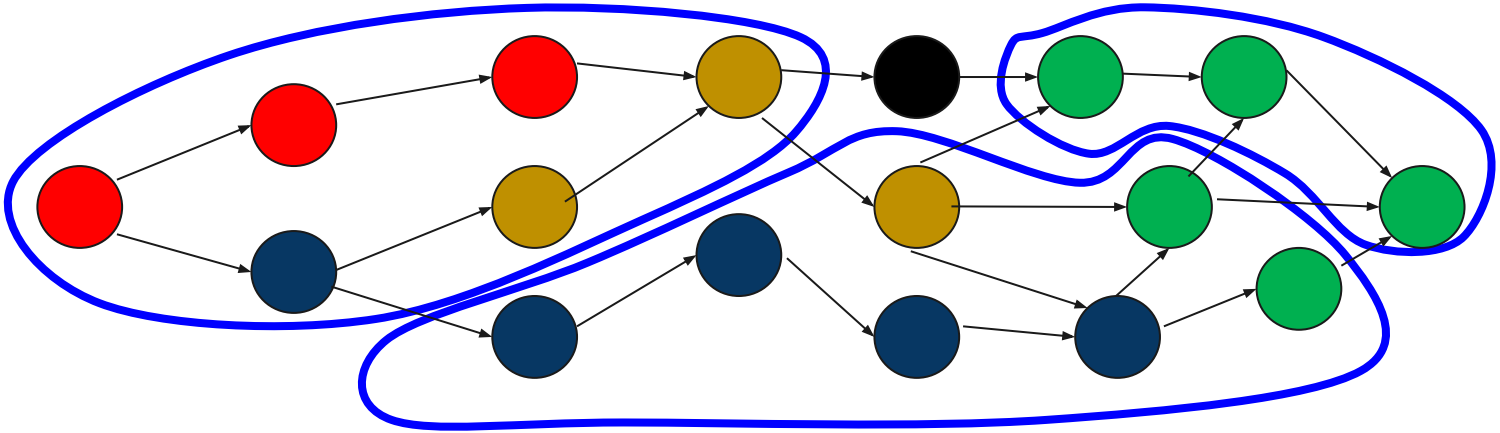
Path gets split among different vertex sets: we recurse on each separately.



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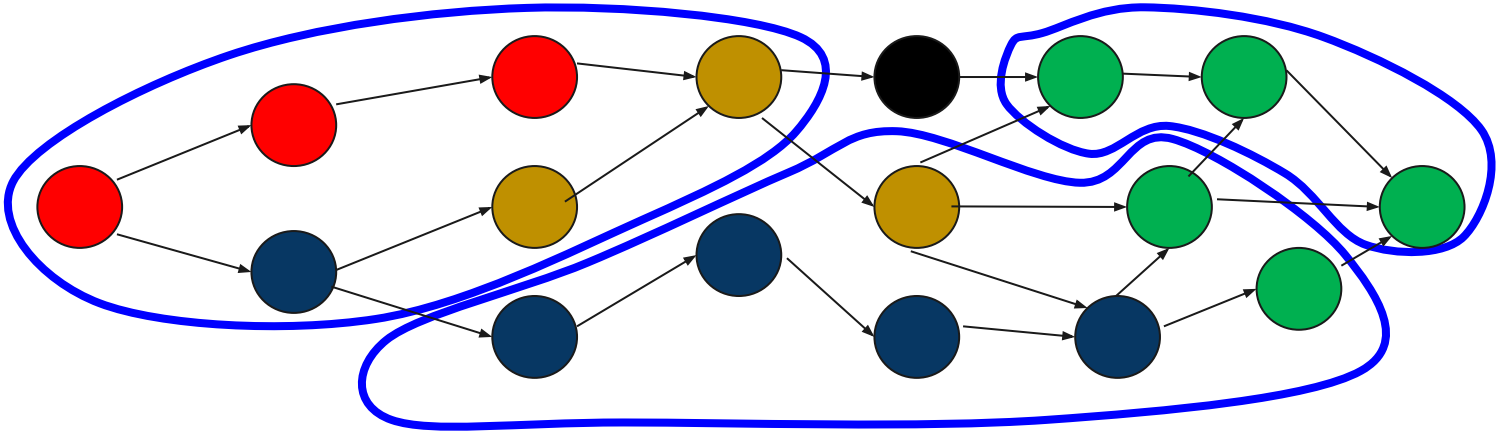
Call **D ∪ A ∪ B** the set of *path-related* vertices. Let $\mathbf{L_P(V) = |D| + |A| + |B|}$.



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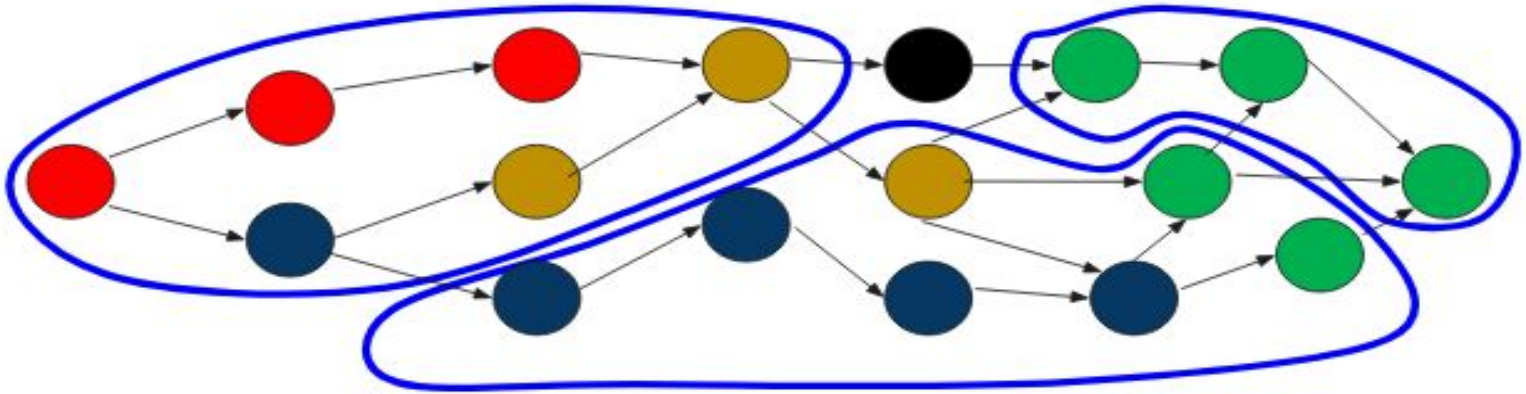
Observation 2: If we chose a shortcutter which is path related for **P,** the path either gets shortcut to length 2 or divided among at most two subproblems.



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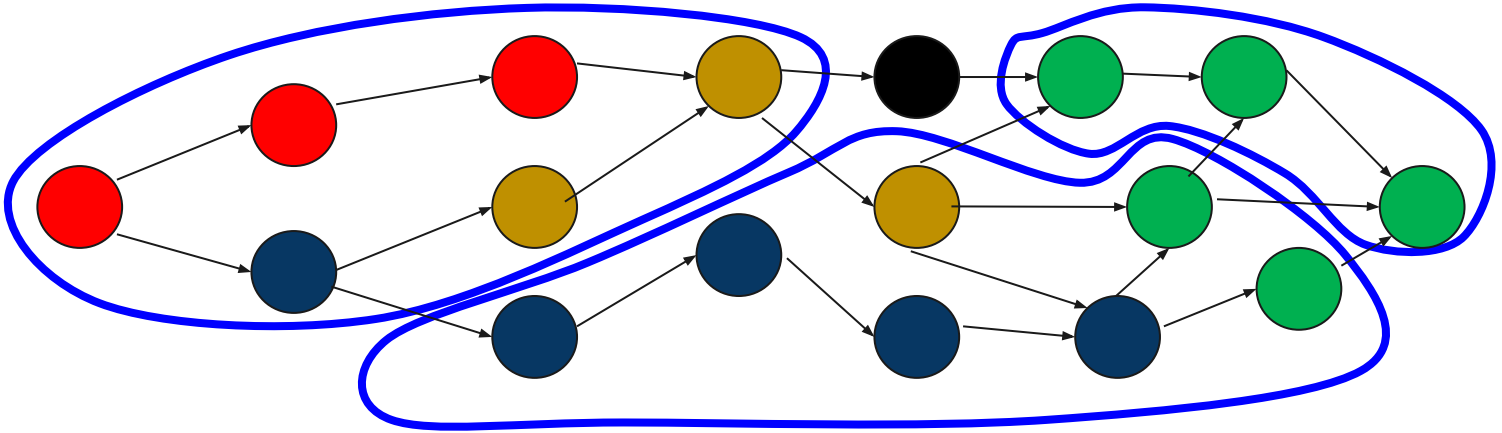
Observe that in the example below, shortcutting means that 3 of the remaining 5 path descendants are separated from any part of the path.



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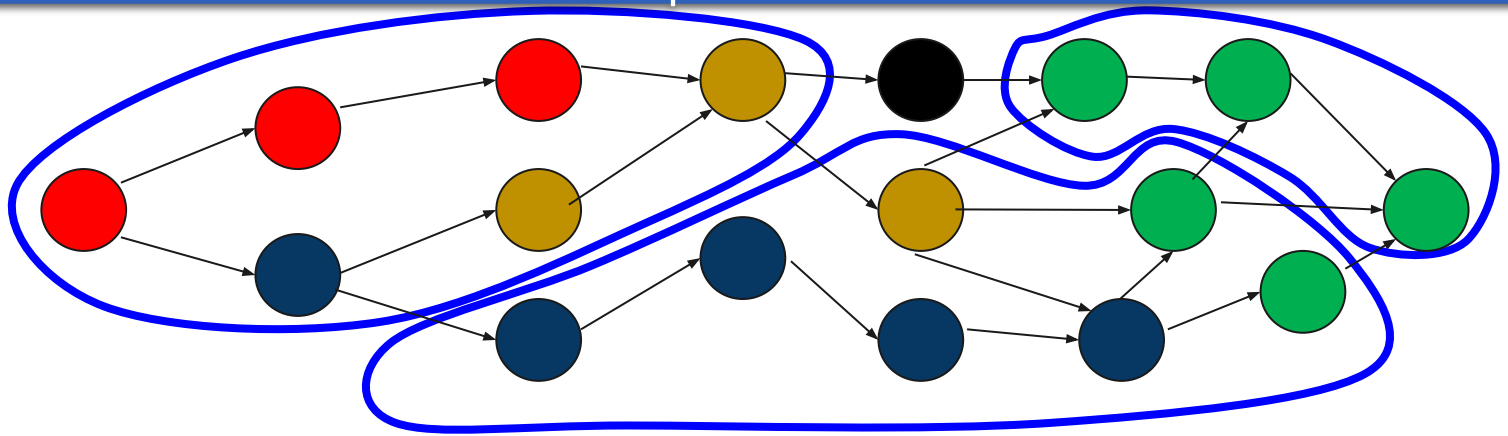
Key Lemma

Conditioned on picking an ancestor (resp. descendant) the expected # of ancestors (resp. descendants) related to any part of P decreases by ½ in expectation.



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Observe **|P| < $L_P(V)$ = |D| + |A| + |B|** < n.

Let $f(L)$ be the expected shortcutted length of a path with L related vertices.

By combining our observations with the above we obtain the recursion

$$f(L) \leq \max_{a+b \leq \frac{3L}{4}} f(a) + f(b) + 1$$

whenever our randomly selected shortcutter is related to **P.**

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$$f(L) \leq \max_{a+b \leq \frac{3L}{4}} f(a) + f(b) + 1$$

As clearly $f(L) \leq L$, running this recursion for *r* levels and optimizing gives

$$f(L_P(V)) \leq f(n) \leq 2^r + (\tfrac{3}{4})^r n \leq \tilde{O}(n^{1/\log_2(8/3)}) \leq \tilde{O}(n^{0.707})$$

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Improved analysis achieves diameter $\tilde{O}(n^{2/3})$.

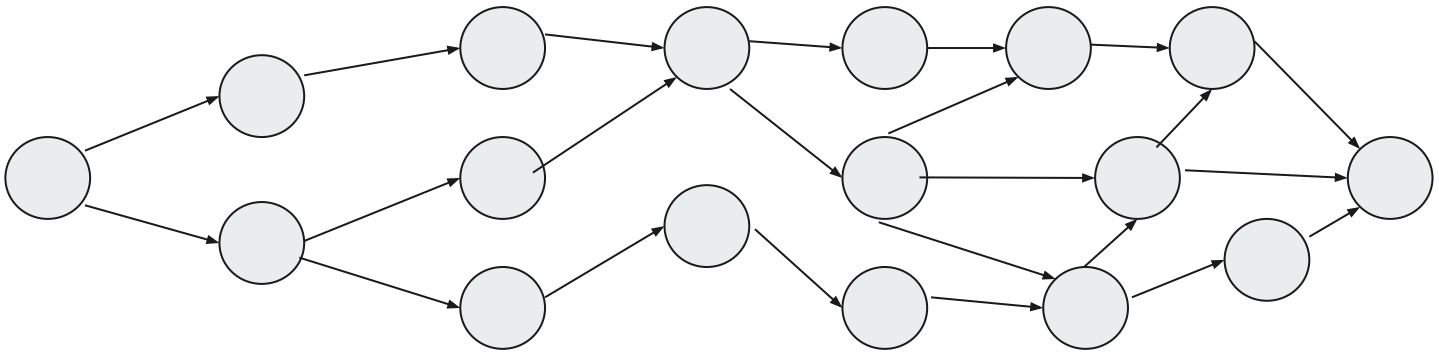
How to improve?

Improved Construction

Pick k random vertices $v_1, v_2, \ldots, v_k$.

Add a shortcut from v_i to every **Descendant** of v_i

Add a shortcut to v_i from every **Ancestor** of v_i



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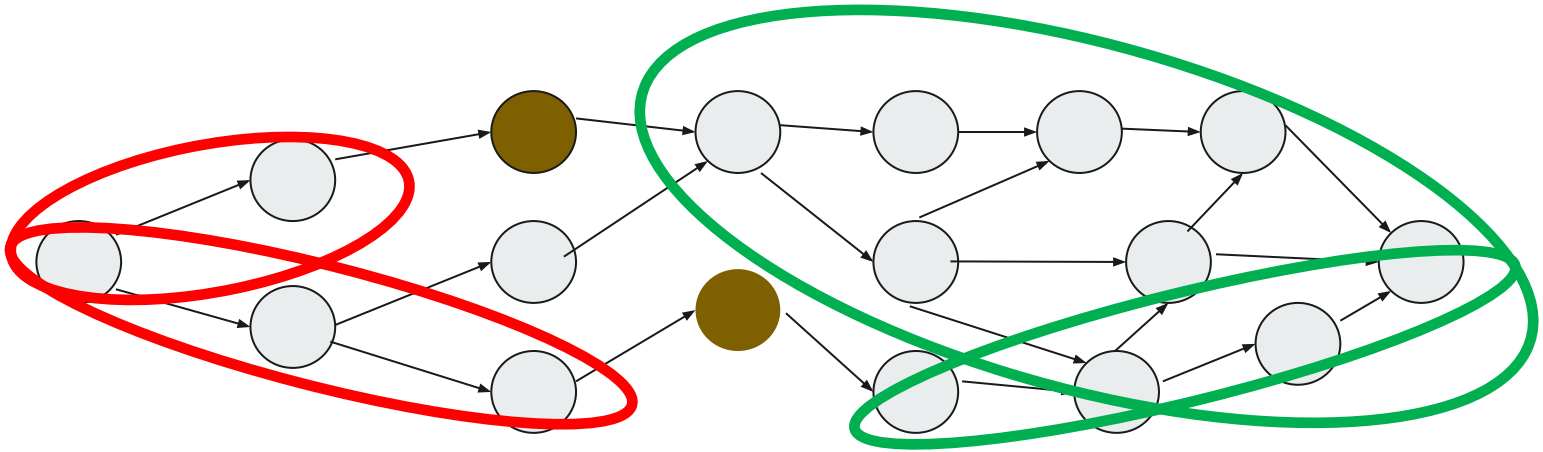

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Partition vertices based on whether they had edges added to/from same vertices.

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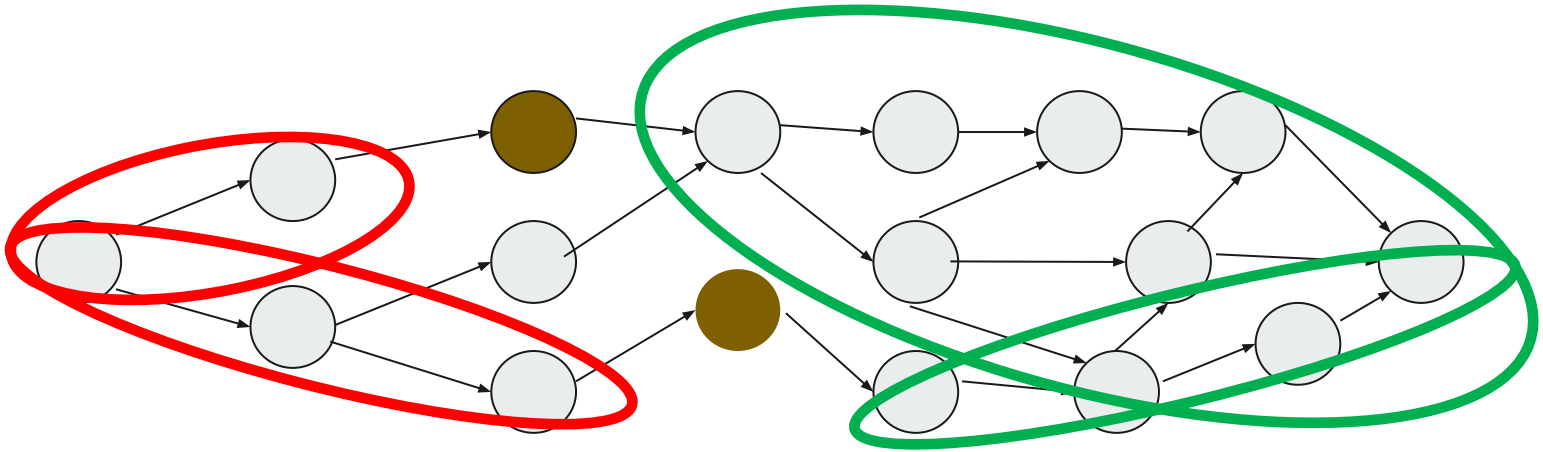

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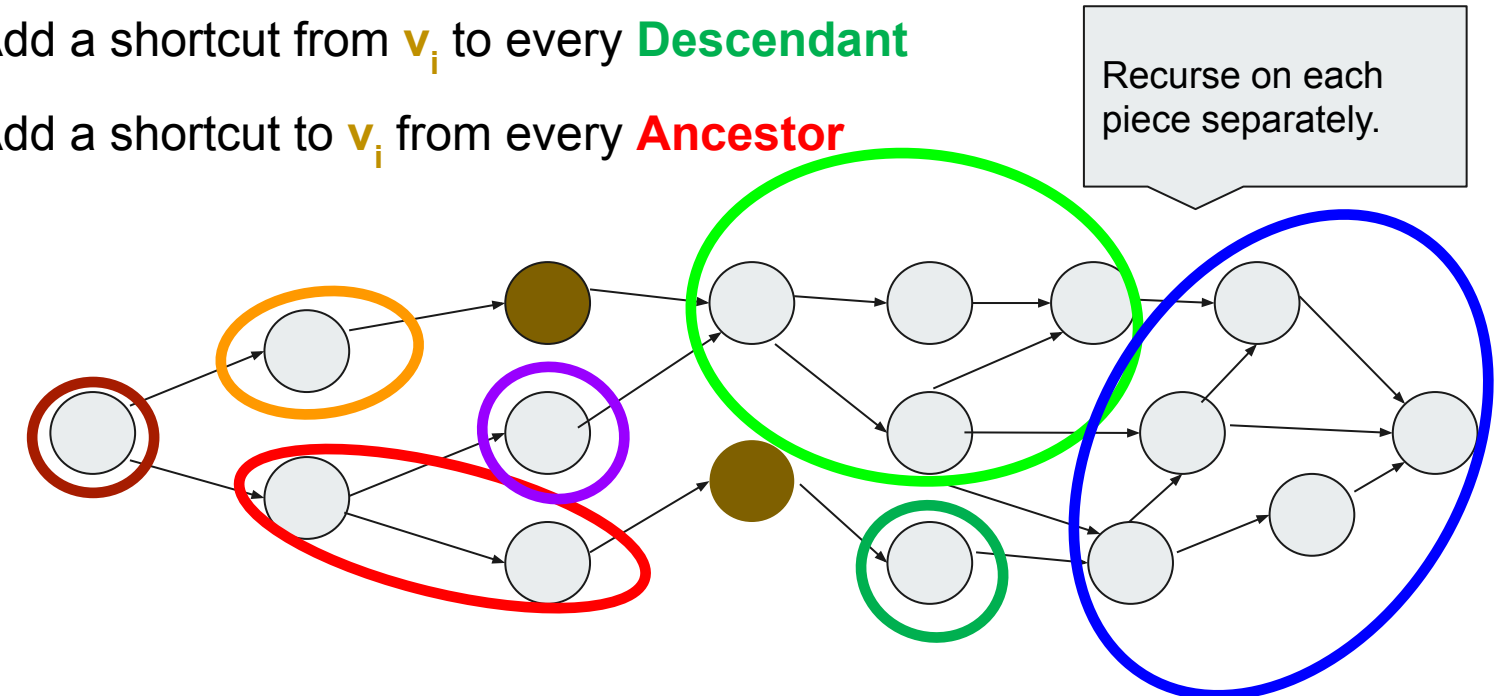

Optimistic Analysis

We start with an incorrect algorithm based on this idea. We can prove the following result:

Optimistic Analysis

Key Lemma Generalization

Conditioned on picking a k ancestors (resp. descendant) the expected # of ancestors (resp. descendants) decrease by 1/(k+1) in expectation.

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Path splits into k+1 pieces each time a path-related vertex is picked.

After **r** levels, total shortcutted path length = $O((k+1)^r) + \left(\frac{2}{k+1}\right)^r n$.

Yields $\tilde{O}(m)$ time and $\tilde{O}(n^{1/2+o(1)})$ diameter!

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How to ensure that we can pick k path related vertices?

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To get similar behavior for this optimistic idea we need to always pick 0 or k path related shortcutters...

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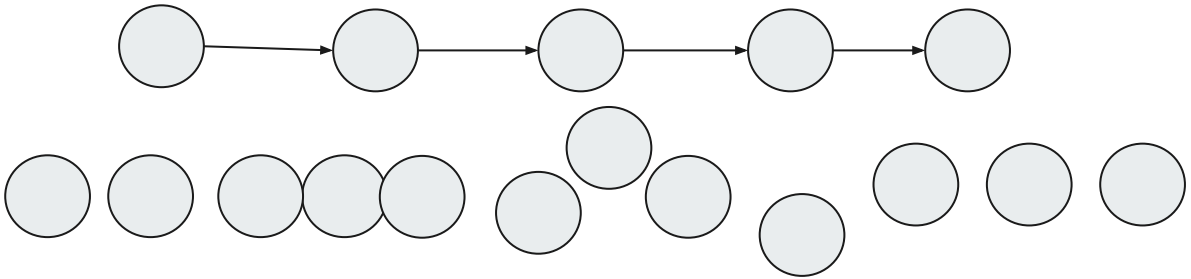

An example bad graph: the “hidden path”. Think length of path is $n^{0.99}$.

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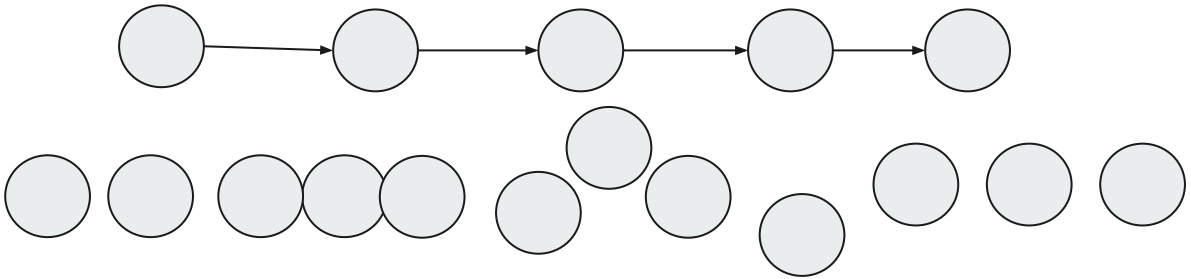

An example bad graph: the “hidden path”. Think length of path is $n^{0.99}$.

If we want to see more than one path-related vertex per round of shortcutting, we need to pick $n^{0.01}$ shortcutters!

Our Algorithm

Is there a way to pick lots of shortcutters in a level of recursion?

Our Algorithm

Key Observation:

As the graph gets partitioned, there are less related pairs of vertices. So when we sample k random vertices, very few are path related in expectation.

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Key Observation Part 2:
As our recursion level deepens, we need to *increase* the number of shortcutter vertices we choose.

Our Algorithm

At recursion depth r, include every vertex into shortcutter set S with probability $\tilde{O}(k^r/n)$.

Shortcut to and from all vertices in S.

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Key Lemma:
With high probability, after round r, every vertex has at most n/k^r total ancestors and descendants.

Total Work:
Thus, total work at depth r+1 is $m \times k^{r+1}/n \times n/k^r = mk$ as desired.

Proving Correctness

We use the generalization of the Key Lemma from before:

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We use a generalization of the key corollary from Fineman:

Key Lemma

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Further, if we pick k path-related shortcutters the path is divided into at most k+1 recursive subproblems.

Proving Correctness

We use a generalization of the key corollary from Fineman:

Key Lemma

Conditioned on picking t path-related shortcutters the expected # of path related vertices decrease by 2/(t+1) in expectation.

Further, if we pick t path-related shortcutters the path is divided into at most t+1 recursive subproblems.

Resulting recursion is more complicated (t is a random variable), but we can prove $\tilde{O}(n^{1/2+o(1)})$ diameter!

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We generalize the parallel implementation in the work of Fineman.

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Goal: reduce the length of a path of length $\tilde{O}(n^{1/2+o(1)})$ by a constant factor in expectation: we can repeat this log number of times to reduce the length of any path.

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Guarantees that there *exists* some copy of a recursively generated subpath that we didnt cut without meaning to.

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First to match the known lower bound for small polynomial hop diameter D.

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Get any sublinear depth, subquadratic work *deterministic* algorithm for reachability?

Questions?