Short Cycles via Low Diameter Decomposition

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Stanford University

Joint work with Sushant Sachdeva, Zejun Yu
University of Toronto

Short Cycle Decomposition

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(k, L) short cycle decomposition of an undirected, unweighted graph G

Decomposition of edges of G into edge disjoint cycles of length <= L and at most k extra edges.

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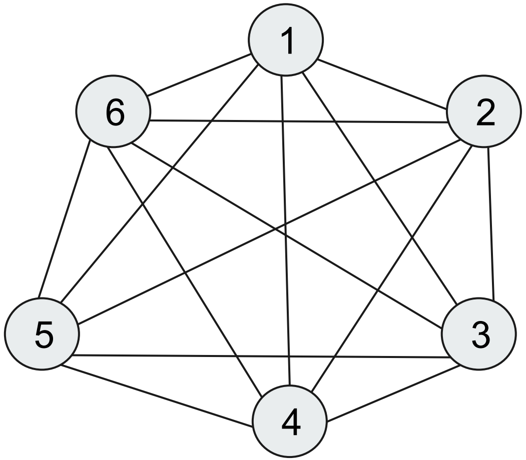

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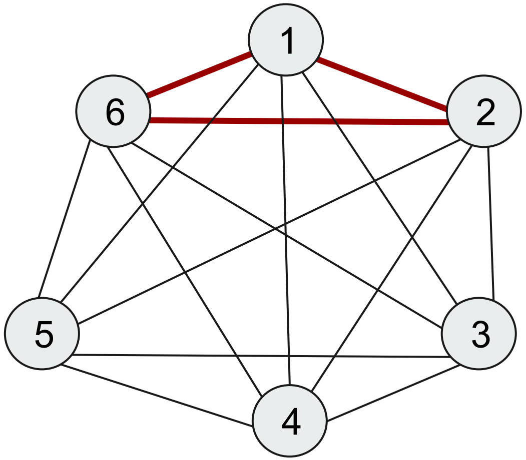

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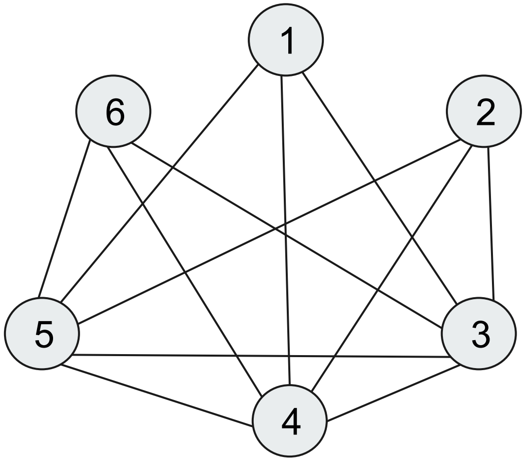

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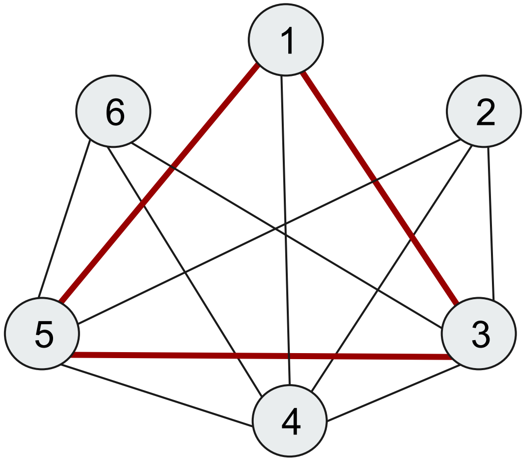

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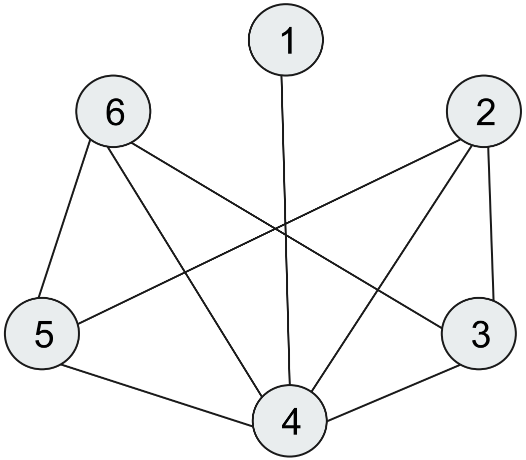

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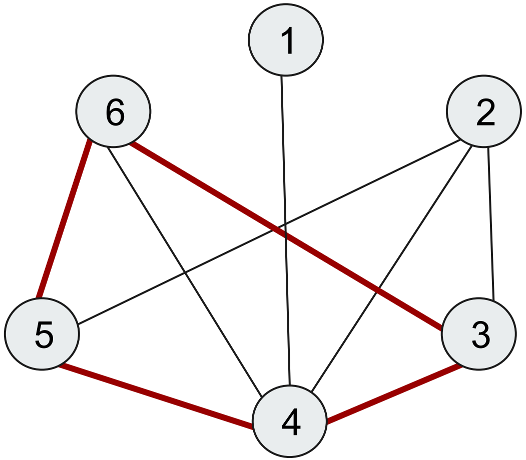

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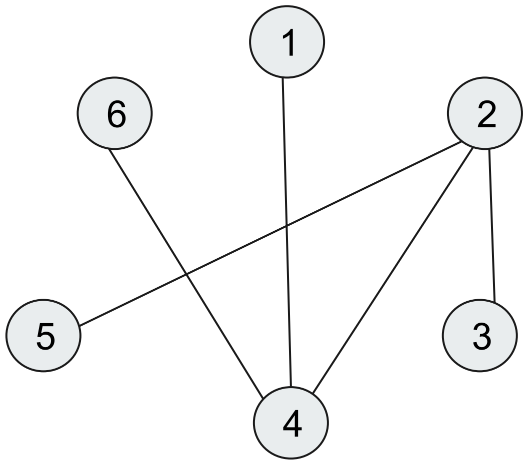

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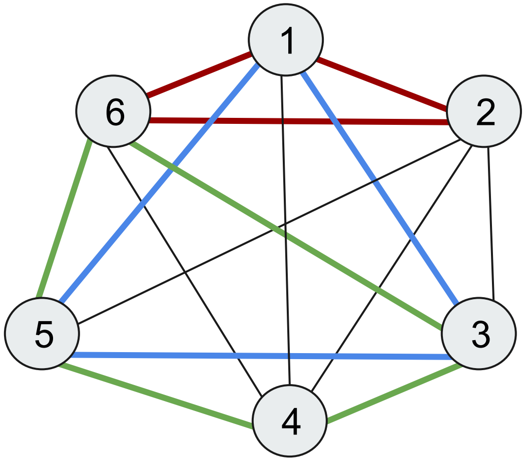

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Approximate some property of a graph G with sparse subgraph H.

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Spanner [Che89]**:** for any pair of vertices u, v we have $d_G(u,v) \leq \alpha \cdot d_H(u,v)$

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Laplacian: $L_G = D_G - A_G$

$$x^T L_G x = \sum_{(u,v)\in E(G)} w_{uv}(x_u - x_v)^2$$

Applications of Spectral Sparsification

Nearly Linear time Laplacian Solvers [ST04, ST14, KMP14, KMP11]

Cut and flow approximation algorithms [She09, She13, CKM+11, KLOS13, Peng16]

Random spanning tree generation [DKP+17]

Estimating determinants + spanning tree counts [DPPR17]

Spectral Sparsification: What's Known

Graph G with n vertices and m edges

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Construction of spectral sparsifier H with $O(n\epsilon^{-2})$ edges [BSS09, BSS12]

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$\Omega(n\epsilon^{-2})$ is optimal, even for arbitrary data structures with cut size queries [BSS12, CKST17]

Spectral Sparsification: new directions

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Spectral Sketches: for an unknown fixed $x \in \mathbb{R}^n$, $x^T L_G x \approx_\epsilon x^T L_H x$ [ACK+16]

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[JS18]: Data structure with $\tilde{O}(n\epsilon^{-1})$ size

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Effective resistance (Reff) is quadratic form wrt Laplacian pseudoinverse

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[DKW15]: Conjecture that H only needs $\tilde{O}(n\epsilon^{-1})$ edges

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[CKP+17]: Applications to Laplacian solvers for directed graphs

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Short Cycle Decomposition was introduced in [CGP+18] to make progress on problems such as the above

Short Cycle Decomposition: What's known

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Theorem [CGP+18]: There is an algorithm running in time $m \cdot \exp(O(\log n)^{\frac{3}{4}})$ which produces a $\left(n \cdot \exp(O(\log n)^{\frac{1}{2}}, \exp(O(\log n)^{\frac{3}{4}})\right)$ short cycle decomposition.

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Extra edges

Cycle length

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Short Cycle Decomposition: Applications [CGP+18]

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Graphical spectral sketches and Resistance Sparsifiers

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Improvements to the short cycle decomposition algorithm in [CGP+18] give immediate improvements for these applications

Our Results

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Theorem [LSY19]: For any constant $\delta \leq \frac{1}{2}$, algorithm running in time $O(mn^{\delta})$ for $\left(O(n), O(\log n)^{\frac{1}{\delta}-1}\right)$ short cycle decomposition

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Uses low diameter decomposition [LS90], instead of expander decomposition (used in [CGP+18])

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First almost linear time Eulerian graph sparsification algorithm *without expander decomposition*.

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Intuition: low depth spanning trees

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Delete vertices until all vertices have degree >= 3

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BFS from anywhere to find a short cycle

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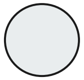

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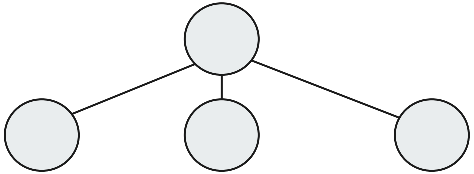

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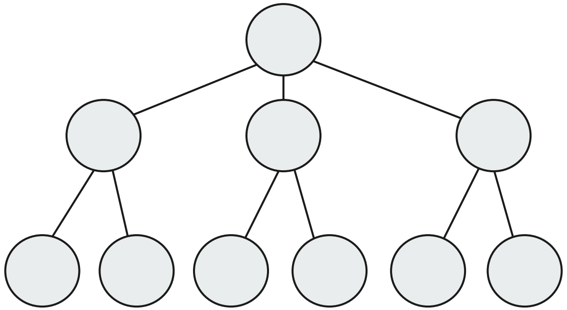

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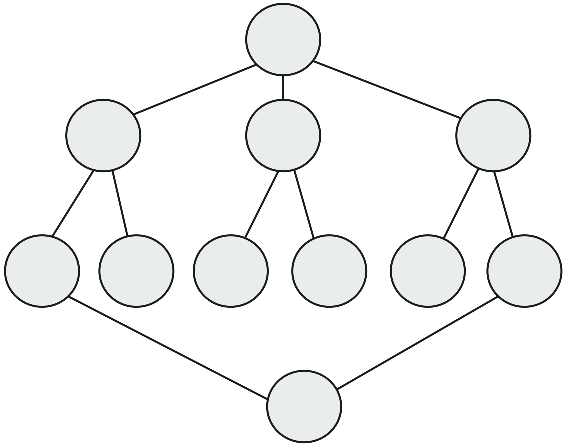

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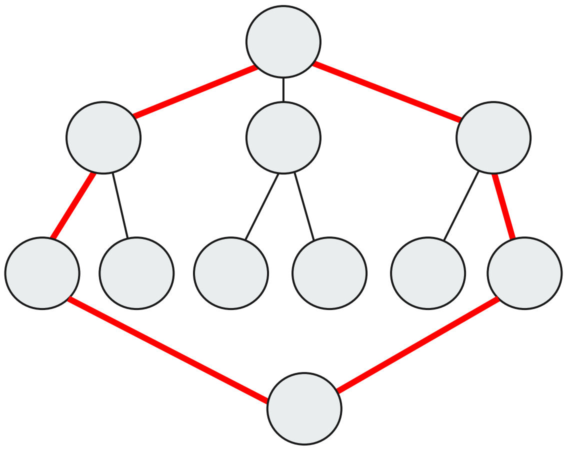

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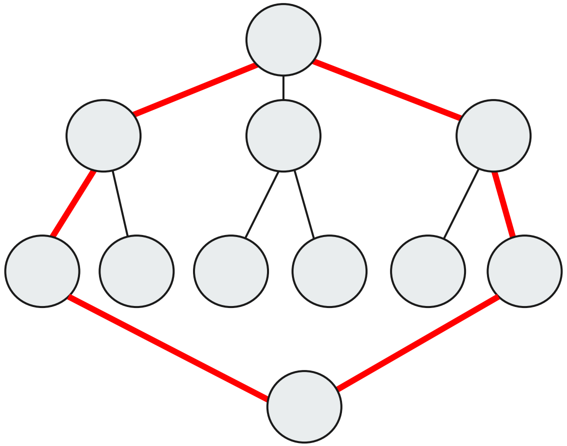
BFS from anywhere to find a short cycle

Delete cycle and repeat



Reduction to sparse, bounded degree graphs

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Graph G with n vertices and m edges

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$m = 10n$

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Maximum degree is bounded (say at most 40).

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Lemma (informal): If we can do $(O(n), O(\log n)^c)$ short cycle decomposition on such graphs efficiently, we can do it on all graphs.

Main Theorem

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G has n vertices, $m = 10n$ edges.

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Maximum degree Δ

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Maximum degree Δ

[LSY19]: For any integer $c \geq 1$ we can find in time $O(500^c m n^{\frac{1}{c+1}})$ *vertex-disjoint* cycles of length $O(\log n)^c$ containing $\frac{m}{10\Delta}$ total vertices.

Preliminaries to algorithm

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Vertex-disjoint analogue of naive cycle decomposition

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In time $O(n^2)$ we can find vertex-disjoint cycles of length $O(\log n)$ containing $\frac{m}{10\Delta}$ vertices.

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Vertex-disjoint analogue of naive cycle decomposition

In time $O(n^{\frac{3}{2}})$ we can find vertex-disjoint cycles of length $O(\log n)$ containing $\frac{m}{10\Delta}$ vertices.

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Low diameter decomposition [LS90] (analogue to low depth tree in naive cycle decomposition)

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Low diameter decomposition [LS90] (analogue to low depth tree in naive cycle decomposition)

[MPX13]: For any parameter β we can remove βm edges to make each remaining connected component have diameter $O(\beta^{-1} \log n)$

Contraction

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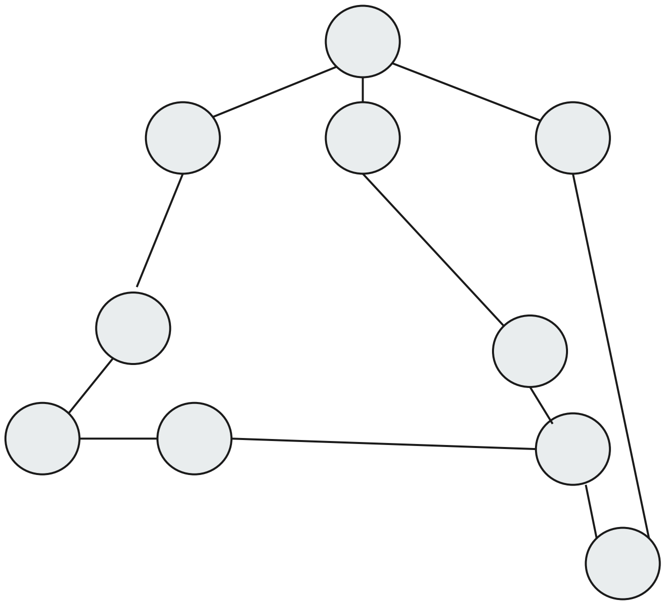

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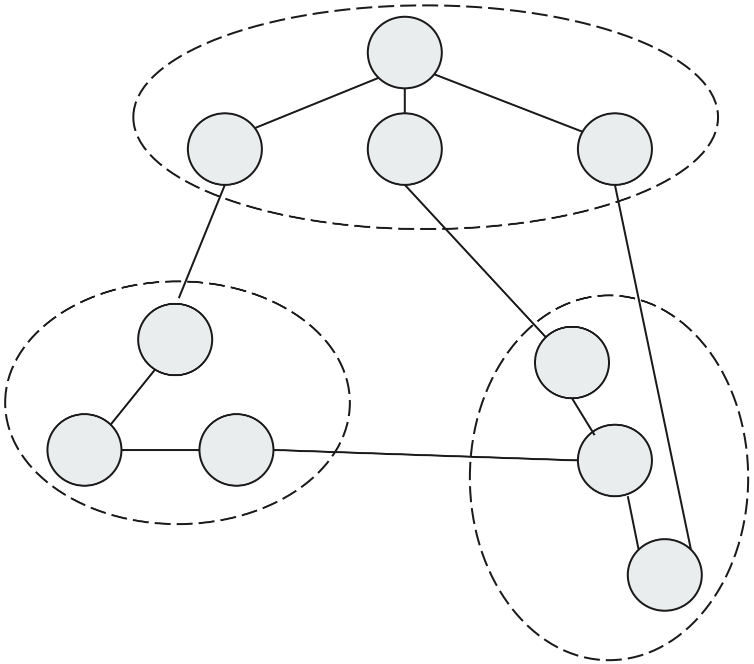

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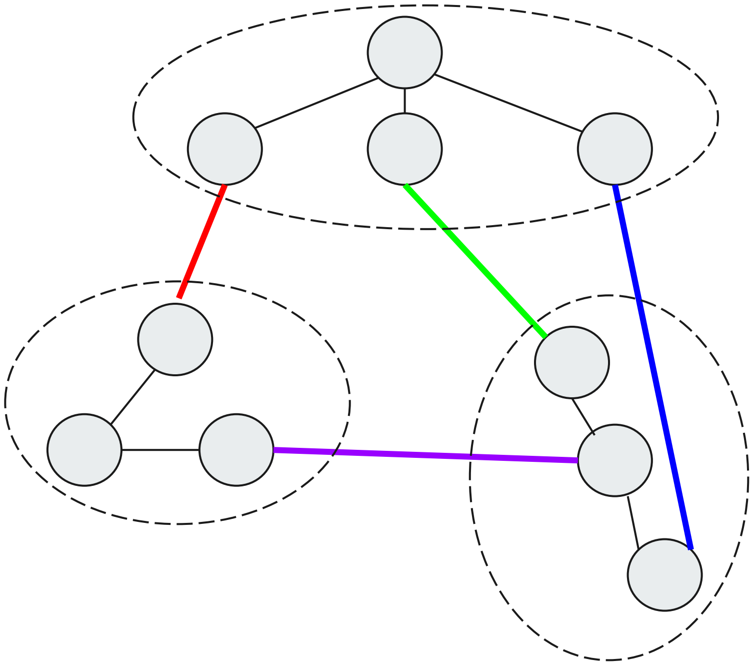

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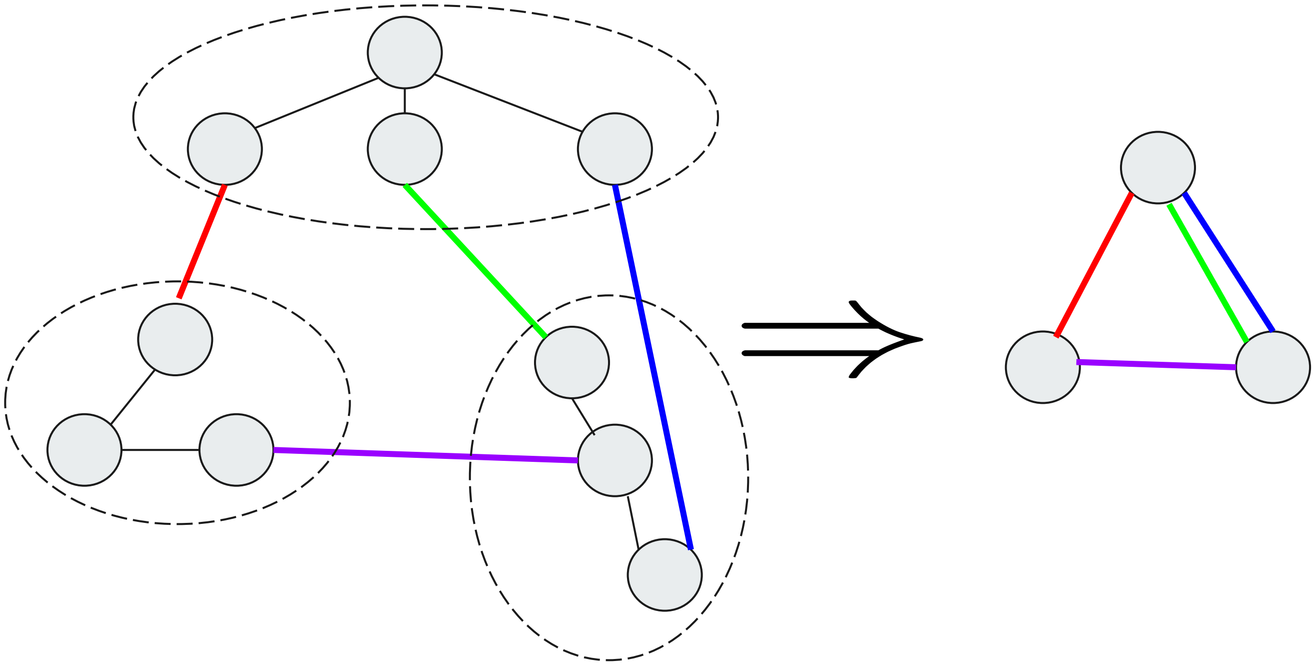

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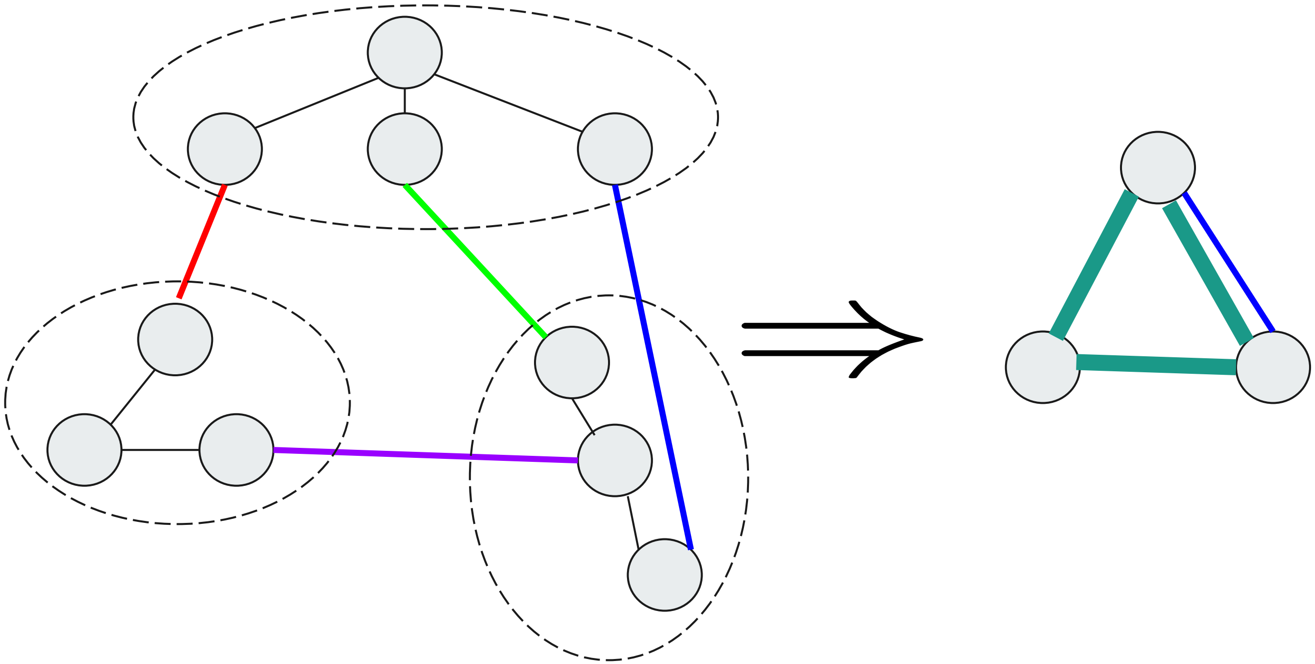

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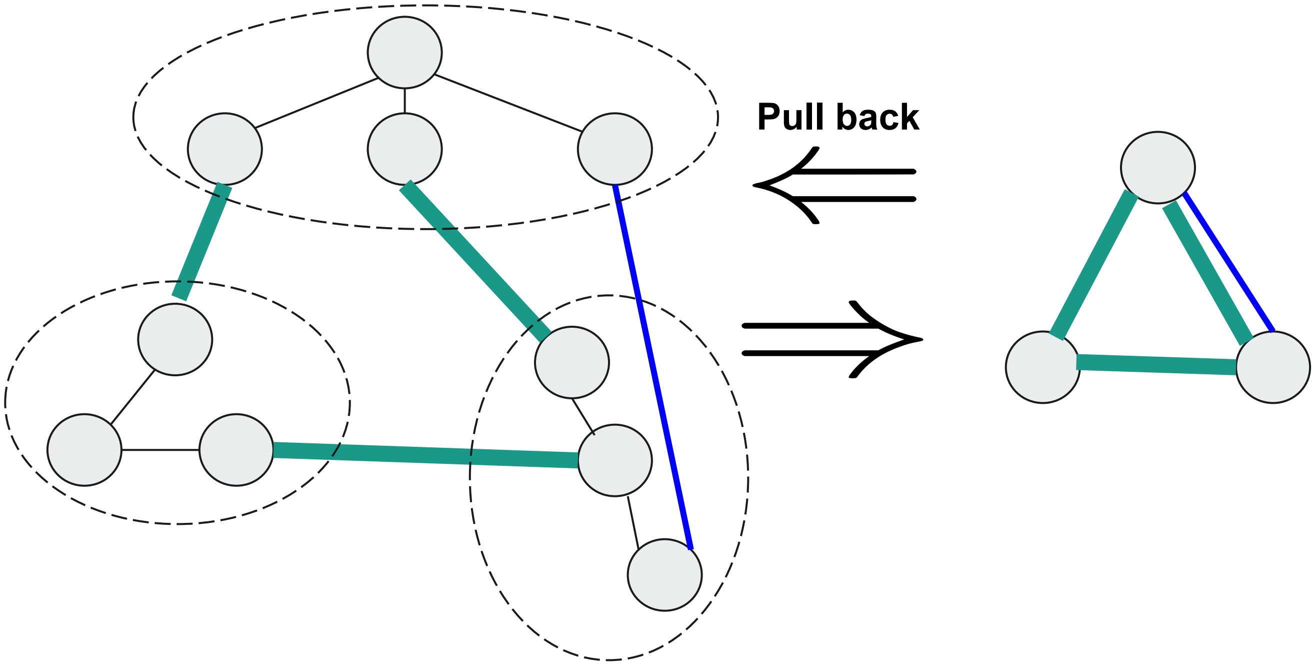

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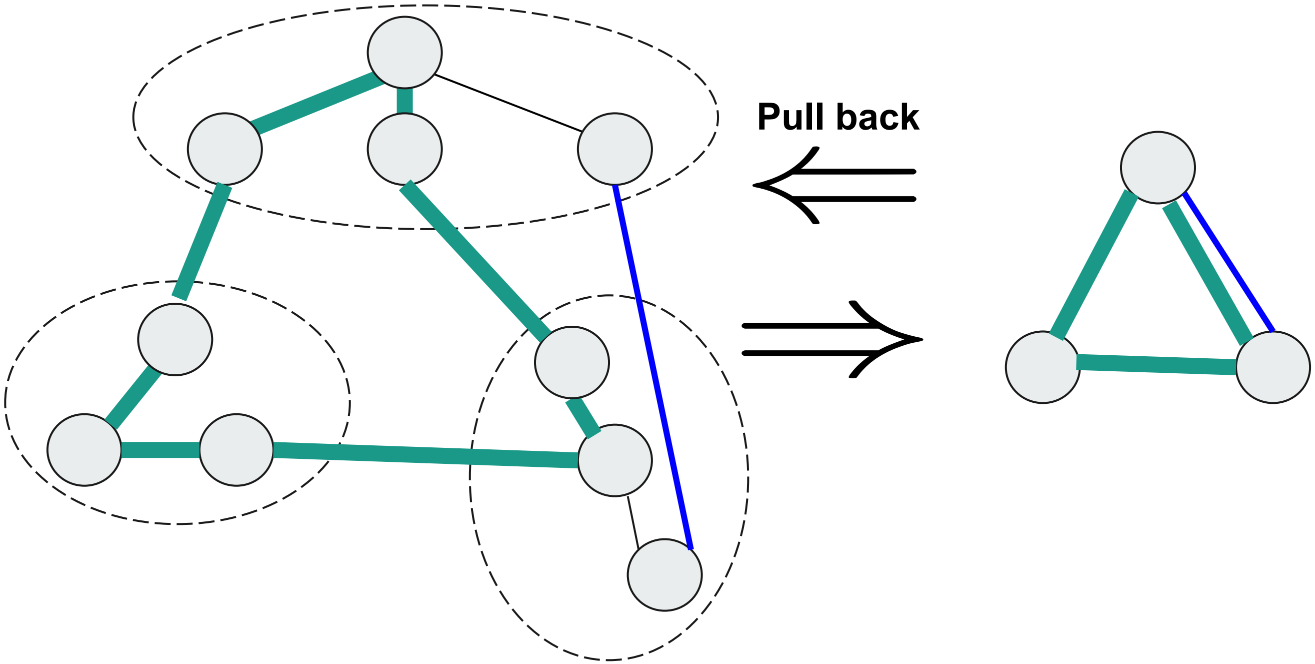

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Partition G into components G_1, G_2, …, G_t, each with diameter $O(\log n)$ and around k vertices

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Contract the components, and recurse on the new graph H (which has $\frac{n}{k}$ vertices)

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Pull cycles in H up to G

Partitioning the vertices

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Do a low diameter decomposition; components might have > k vertices

Build a low diameter spanning tree on each component

Partitioning the vertices

Partition G into components G_1, G_2, ..., G_t, each with diameter $O(\log n)$ and around k vertices

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Do a low diameter decomposition; components might have > k vertices

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Partition the spanning tree into components of size approximately k

Partitioning the vertices

Partition G into components G_1, G_2, ..., G_t, each with diameter $O(\log n)$ and around k vertices

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Do a low diameter decomposition; components might have > k vertices

Build a low diameter spanning tree on each component

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Works because graph has bounded degree

Recursion details

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Fix: Recall that we are looking for cycles containing $\frac{m}{10\Delta}$ vertices

Sparsify H, proportionally reducing m and Δ

This way, the maximum degree of H isn’t much larger than that of G

Conclusion

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We give new algorithms for short cycle decomposition

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Improvements to all of:

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Estimating effective resistances

Degree-preserving sparsifiers and Eulerian directed graph sparsifiers

Recent work

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Parter and Yogev have an upcoming result which gets $(O(n \log n), O(\log^2 n))$ short cycle decomposition in time $n^{1+o(1)}$.

Thanks for listening!