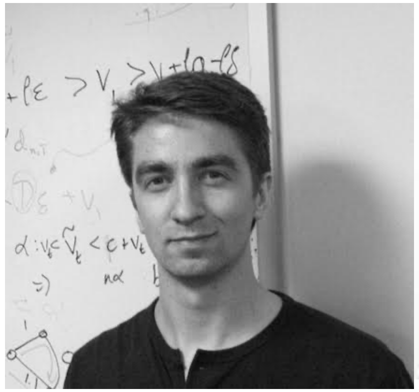
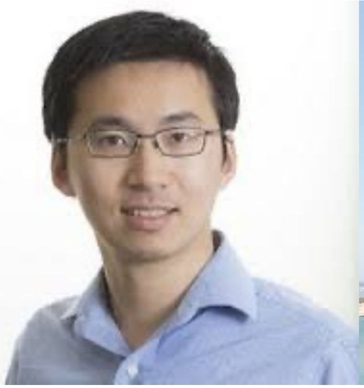
Maximum Flow and Minimum-Cost Flow in Almost Linear Time

Li Chen (Georgia Tech), **Yang P. Liu** (Stanford University)
Joint with Rasmus Kyng, Richard Peng, Maximilian Probst Gutenberg, Sushant Sachdeva

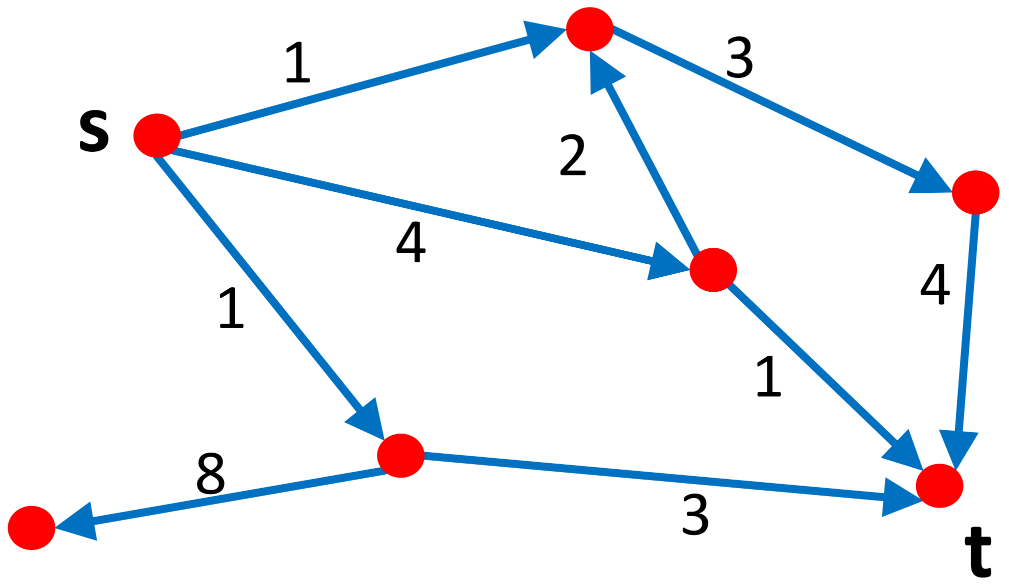
Talk Outline

Part 1: Problem History and Our Results

Part 2: Using Tree Embeddings to Find Approx. Min-Ratio Cycles

Maximum Flow Problem

- Directed graph $G = (V, E)$ source s, sink t, *capacities* $u_e \geq 0$.
- m edges, n vertices, maximum capacity U.



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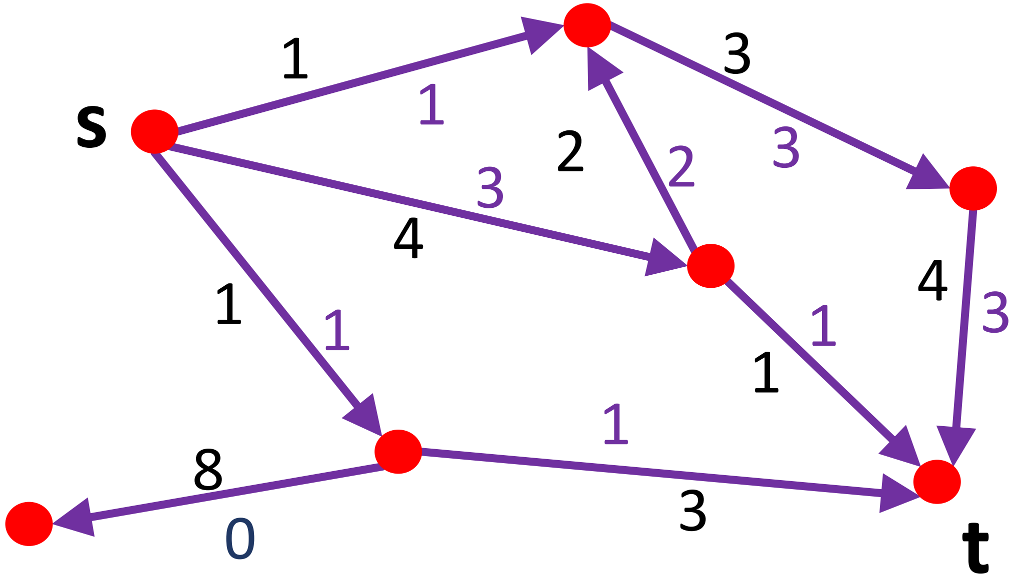

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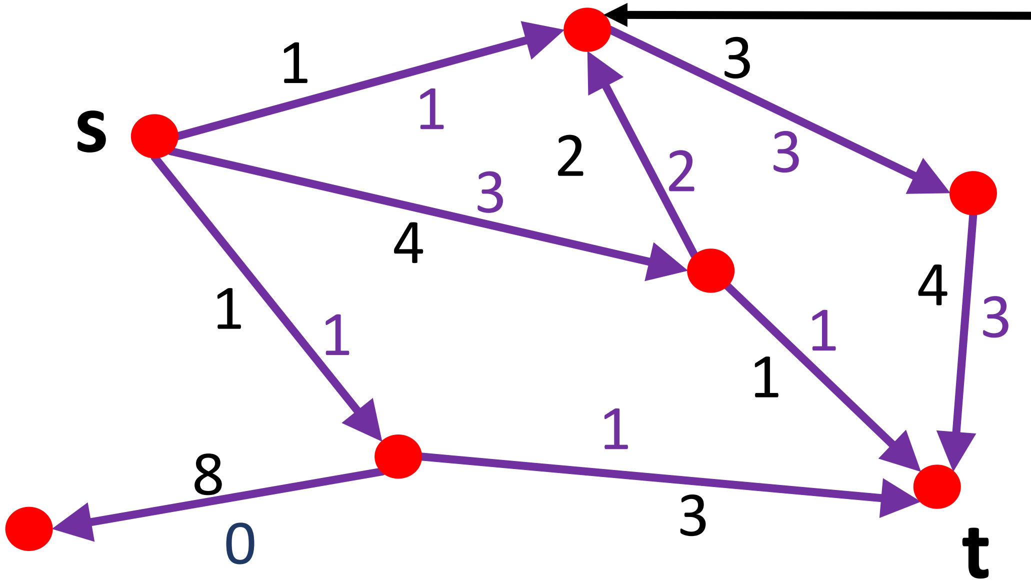

Demand constraint: all vertices except s, t have equal incoming/outgoing flow

Total flow: number of units leaving s / entering t

Capacity constraint: amount of flow on edge e in [0, u_e]

Why Study Flows?

- **Graph opt:** Flows are a broad class of graph optimization problems.
- Route 1 unit from s to t while minimizing some *cost* given by convex functions on edges, $\sum_{\mathrm{edges}\ e} \mathrm{cost}_{\mathrm{e}}(f_{\mathrm{e}})$.
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Our results cover all the problems mentioned above.

- **Direct applications:** Bipartite matching, densest subgraph, Gomory-Hu trees / connectivity, negative-weight shortest path

$\tilde{O}(.)$ hides poly(log(mnU)) factors

Comparison of Previous Algorithms

	Runtime	Iterations	Amortized Cost
[GN80, ST83]	$O(mn \log n)$	$\tilde{O}(n)$	$\tilde{O}(m)$
[GR98]	$\tilde{O}(m^{3/2})$	$\tilde{O}(m^{1/2})$	$\tilde{O}(m)$
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[K**L**S20]	$m^{4/3+o(1)}U^{1/3}$	$m^{1/3+o(1)}U^{1/3}$	$m^{1+o(1)}$
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[BLNPSSSW20, BL**L**SSSW21]	$\tilde{O}(m + n^{3/2})$	$\tilde{O}(n^{1/2})$	$\tilde{O}(m/n^{1/2} + n)$
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Augmenting paths / blocking flows and dynamic tree data structure.

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Blocking flows, capacity scaling, and much more.

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Interior point method (IPM) + Laplacian solver: also min-cost.

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IPM + Laplacian + weight changes to reduce iterations.

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IPM + p-norm flows: more sophisticated iteration reduction

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IPM + generic iteration count reduction for all linear programs

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IPM + dynamic electrical flows: heavy-hitters, sparsification

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IPM + dynamic electrical flows via random walk + Schur complement

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$\tilde{O}(\boldsymbol{m}/\boldsymbol{\epsilon})$ for $(1+\epsilon)$-approximate *undirected* max-flow [S13,KLOS14,P16,BGS22]

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- **Combinatorial algorithms** [GN80, ST83, GT87]
  - Augmenting paths, shortest paths, blocking flows.
  - Cycle cancelling by finding *minimum-mean cycles.*
  - Work primarily with residual graphs that are *directed*.
- **Continuous optimization** [DS08, M13/16, LS14, etc.]
  - Augmenting *electrical flows* (ℓ_2 primitive) or *circulations*
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Optimization Method Result: Reduction to Slowly Changing Min-Ratio Cycle Instances

- (Informal Theorem) We compute a maxflow in $m^{1+o(1)}$ iterations of:
- Add circulation c which is $m^{o(1)}$-approx. minimizer to $\langle g, c\rangle / ||Lc||_1$ over circulations, for dynamically changing gradients g, lengths L.
- Coordinates of g, L change at most $m^{1+o(1)}$ times total $m^{1+o(1)}$

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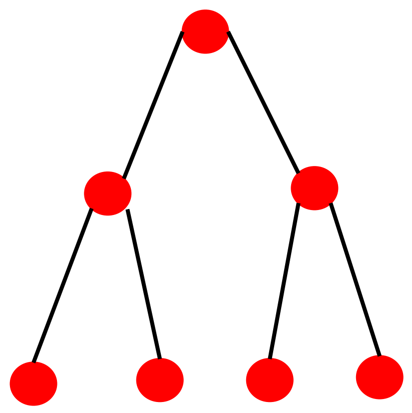

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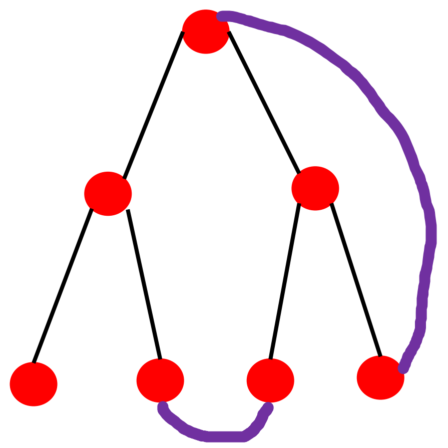

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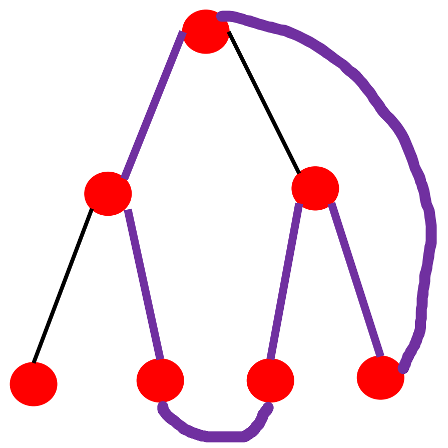

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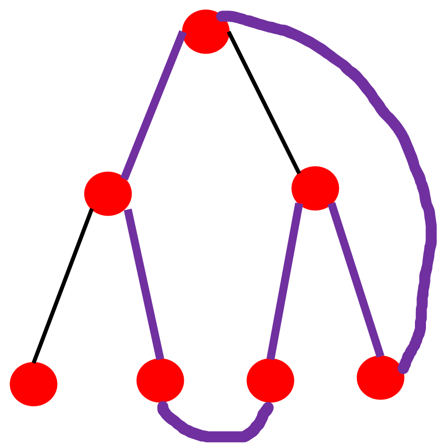
- Simpler version given in [Wallacher-Zimmerman, Math. Prog. '92]



Dynamically Finding Approx. Min-Ratio Cycles

- (Informal theorem) Randomized data structure that supports:
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Framework goes beyond the standard oblivious/adaptive split.

Talk Outline

Part 1: Problem History and Our Results

Part 2: Using Tree Embeddings to Find Approx. Min-Ratio Cycles

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Part 2: Using Tree Embeddings to Find Approx. Min-Ratio Cycles

Algorithm Outline

(Input): Graph G = (V, E) with cap. u_e for each edge e, initial flow $f^{(0)}$

1. For t = 1, 2, …, $m^{1+o(1)}$ iterations:
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Question 1: How to decide what $g^{(t)}, L^{(t)}$ are? How to decide when to update them?

Question 2: How to maintain the tree T and find the cycle c as $g^{(t)}, L^{(t)}$ change?

Potential Reduction Interior Point Method

- Add an undirected edge e*, implicitly directed from (t,s).
- Let F = the maxflow between (s, t)
- Potential function [Kar84]: $\min_{\text{circulation } f} \Phi(f)$, where:
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- Trades off routing more flow (closer to F), and not getting close to capacity constraints.

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- Reduces potential by $m^{-o(1)}$ per iteration, so $m^{1+o(1)}$ total iters.
- If f^* is the maxflow, choosing $c = f^* - f$ is a good direction.

Approx. MRC via Random Tree Embeddings

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- Let c* be the optimal minimizer of $\langle g, c\rangle / ||Lc||_1$.
- Then $\mathbb{E}_T\left[\sum_{\mathrm{edges}\ e} |c_e^*| \,\mathrm{len}(c_T(e))\ \right] \leq \tilde{O}(1) ||Lc^*||_1$

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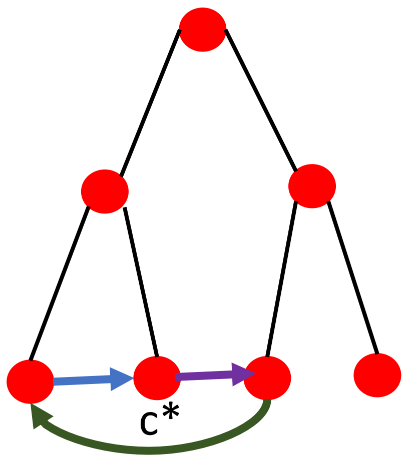

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