CS 15-759: Homework 5

Due: 4/4/2025, 11:59 PM on Canvas

Bonus points: If you find typos in my homework or lecture notes, please email me. You will earn +1 bonus points per typo found, and potentially more for especially egregious typos.

Hints: Hints are on the last page. It is recommended to think about the problem without hints for a while, and then look at the hints when stuck. The problems are meant to be difficult, so there is no shame in looking at the hints. If you make partial progress on problems (e.g., by following the hints) you will get partial points.

Problem 1: Example Self-Concordant Functions (30 Points)

For this problem, use the following definition of ν-self-concordance. For a convex set $K \subseteq \mathbb{R}^n$, the function $\Phi : K \to \mathbb{R} \cup \{+\infty\}$ is ν-self-concordant if:

1. For all $x \in K$, it holds that $\nabla\Phi(x)^\top \nabla^2\Phi(x)^{-1}\nabla\Phi(x) \le \nu$, and
2. For all $x \in K$ and $u, v, w \in \mathbb{R}^n$ it holds that

$$\left|\nabla^3\Phi(x)[u, v, w]\right| \le 2\|u\|_{\nabla^2\Phi(x)}\|v\|_{\nabla^2\Phi(x)}\|w\|_{\nabla^2\Phi(x)}.$$

Here, $\|x\|_M := \sqrt{x^\top Mx}$ is the matrix norm, and $\nabla^3\Phi(x)[u, v, w]$ is the third order directional derivative in directions u, v, w.

Prove that the following functions are self-concordant:

1. $\Phi(x) = -\log x$ on the set $K = \{x : x \ge 0\} \subseteq \mathbb{R}$ if 1-self-concordant.
2. $\Phi(x) = -\log(1 - \|x\|_2^2)$ on the set $K = \{\|x\|_2 \le 1 : x \in \mathbb{R}^n\}$ is 1-self-concordant.
3. $\Phi(X) = -\log\det X$ on the space of $n \times n$ PSD matrices is n-self-concordant.

Problem 2: Sums of Self-Concordant Functions (25 Points)

Let $\Phi_1, \Phi_2, \dots, \Phi_k$ be $\nu_1, \dots, \nu_k$-self-concordant functions on a domain K. Prove that $\Phi_1 + \cdots + \Phi_k$ is $\nu_1 + \cdots + \nu_k$-self-concordant.

Problem 3: Optimality Gap (25 Points)

Define $x^* := \text{argmin}_{Ax=b,x\ge0} c^\top x$ where $A \in \mathbb{R}^{d\times n}$. Let $xs = \mu, Ax = b, s = c - A^\top y$ be a point on the central path. Prove that $c^\top(x - x^*) \le n\mu$.

Problem 4: Hessians are Locally Similar (20 Points)

Prove that if Φ is a self-concordant function (see the definition in Problem 1), then for x and $\|\Delta\|_{\nabla^2\Phi(x)} \leq \varepsilon < 1$ that

$$(1-\varepsilon)\nabla^2\Phi(x) \preceq \nabla^2\Phi(x+\Delta) \preceq \frac{1}{1-\varepsilon}\nabla^2\Phi(x).$$

Note: Feel free to provide a weaker bound (no points will be deducted). For example, feel free to assume that $\varepsilon < 1/1000$ is sufficiently small, and you prove a bound such as

$$(1-10\varepsilon)\nabla^2\Phi(x) \preceq \nabla^2\Phi(x+\Delta) \preceq (1+10\varepsilon)\nabla^2\Phi(x).$$

1 Hints

Problem 1: You can't do anything tricky here other than to do the calculations.

Problem 2: Keep using the triangle inequality and Cauchy-Schwarz.

Problem 3: Start with $x^\top s = n\mu$. You eventually need to use that for any y with $A^\top y \leq c$ that $b^\top y \geq c^\top x^*$ (weak duality).

Problem 4: We want to prove that $v^\top \nabla^2 \Phi(x + \Delta) v \leq \frac{1}{1-\varepsilon} v^\top \nabla^2 \Phi(x) v$. Define

$$f(t) = v^\top \nabla^2 \Phi(x + t\Delta) v$$

and use that $f'(t) = \nabla^3 \Phi(x + t\Delta)[\Delta, v, v]$ and use property 1 of self-concordance.

References