CS 15-759: Homework 1

Due: 1/24/2025, 11:59 PM on Canvas

Bonus points: If you find typos in my homework or lecture notes, please email me. You will earn +1 bonus points per typo found, and potentially more for especially egregious typos.

Hints: Hints are on the last page. It is recommended to think about the problem without hints for a while, and then look at the hints when stuck. The problems are meant to be difficult, so there is no shame in looking at the hints. If you make partial progress on problems (e.g., by following the hints) you will get partial points.

Problem 0: Linear Algebra Facts

Let A, B be PSD matrices. We say that $A \preceq B$ if for all x, it holds that $x^\top Ax \le x^\top Bx$.

Spectral theorem: For any symmetric real matrix A, we can write $A = \sum_{i=1}^n \lambda_i v_i v_i^\top$ for some mutually orthogonal unit vectors $v_1, \dots, v_n$ and real eigenvalues $\lambda_1, \dots, \lambda_n$.

A symmetric matrix A is PSD iff all its eigenvalues are nonnegative.

For a PSD matrix A, we define its square root as $A^{1/2} := \sum_{i=1}^n \lambda_i^{1/2} v_i v_i^\top$. It is not hard to check that $A^{1/2}$ is PSD and $(A^{1/2})^2 = A$. Similarly, $e^A := \sum_{i=1}^n e^{\lambda_i} v_i v_i^\top$.

Problem 1: PSD Matrix Inequalities (35 Points)

Prove the following facts.

(a) Let A, B be PSD. Prove that if $A \preceq B$ then for a matrix M that $M^\top AM \preceq M^\top BM$.

(b) Let A, B be positive definite matrices satisfying $A \preceq B$. Prove that $B^{-1} \preceq A^{-1}$.

(c) Let A, B be positive definite matrices satisfying $A \preceq B$. Prove that $A^{1/2} \preceq B^{1/2}$.

(d) Find positive definite matrices $A \preceq B$ but $A^2 \not\preceq B^2$.

(e) Find positive definition matrices $A \preceq B$ but $e^A \not\preceq e^B$.

(f) Let A, B be PSD matrices. Prove that the matrix whose (i, j) entry is $a_{ij}b_{ij}$ is PSD.

(g) (Laplacians are PSD) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix such that for all $i = 1, \dots, n$ it holds that $A_{ii} \ge \sum_{j\ne i} |A_{ij}|$. Then A is PSD.

Problem 2: Power Method (20 Points)

This problem is useful for Problem 3. Let $\mathsf{nnz}(v)$ is the number of nonzero entries in the vector v (or matrix).

Solve: Let A be a PSD matrix whose largest eigenvalue is λ and second eigenvalue is $\lambda_2 \leq (1-\delta)\lambda$. Let v be the top eigenvector of A, with $\|v\|_2 = 1$. Give an algorithm that with high probability finds a vector w with $\|w\|_2 = 1$ satisfying $\|v - w\|_2 \leq \varepsilon$ in time $\widetilde{O}(\mathsf{nnz}(A)\delta^{-1} \log(1/\varepsilon))$.

Problem 3: Solving a Special Linear System (45 Points)

This problem is adapted from [1].

Let $n, m > 0$ be positive integers, and $\delta > 0$. You should think of δ as being very small, eg. $\delta = (nm)^{-100}$. Consider the matrix $A = \frac{1}{m}\sum_{i=1}^{m}(I - v_i v_i^\top)$ where $v_1, \ldots, v_m \in \mathbb{R}^n$ are vectors satisfying $\|v_i\|_2 \leq 1 - \delta$. Finally, let $N = n + \sum_{i=1}^{m} \mathsf{nnz}(v_i)$.

The goal of this problem will be to give a (randomized) near-linear time algorithm that solves linear systems in A. In other words, given an input vector $z \in \mathbb{R}^n$ with $\|z\|_2 = 1$, the algorithm will output some y with $\|Ay - z\|_2 \leq \varepsilon$, with runtime $O(N \log(Nm(\delta\varepsilon)^{-1})^{O(1)})$.

(a) Prove that A is positive definite.

(b) Given an algorithm running in time $O(N \log(Nm(\delta\varepsilon)^{-1})^{O(1)})$ that given A and z with $\|z\|_2 = 1$ with high probability finds y with $\|Ay - z\|_2 \leq \varepsilon$.

1 Hints

Problem 1(b): First prove the statement when $B = I$, and then reduce to this case.

Problem 1(c): First prove the following statement: if X, Y are symmetric matrices satisfying that Y is positive definite, and $XY + YX$ is PSD, then X is PSD.

Problem 1(d), 1(e): There are 2×2 examples.

Problem 1(f): Write A and B in terms of their spectral decomposition via the spectral theorem. This effectively reduces the problem to the case where A and B are rank 1.

Problem 1(g): Write $x^\top Ax$ as the sum of squares.

Problem 2: Let g be a random Gaussian vector and compute $A^k g$ for sufficiently large k. To prove this works, decompose g along the eigenvectors of A.

Problem 3(b):

1. Give a near-linear time algorithm for finding $\|Ay - z\|_2 \le \varepsilon$ in the case that $\lambda_{\min}(A) \ge 0.1$.
2. Prove that if A has an eigenvalue λ satisfying $\lambda \le 0.1$, then all other eigenvalues of A satisfy are at least 0.2.
3. Use the power method to find the minimum eigenvector of A to high-accuracy in this case.

References

[1] Michael B. Cohen, Yin Tat Lee, Gary L. Miller, Jakub Pachocki, and Aaron Sidford. Geometric median in nearly linear time. In Daniel Wichs and Yishay Mansour, editors, *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2016, Cambridge, MA, USA, June 18-21, 2016*, pages 9–21. ACM, 2016.